Success in Contests[1]

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first version: November 15, 2019

Abstract

Models of two contestants exerting effort to win a prize are very common and widely used in political economy. The contest success function plays as fundamental a role in the theory of contests as does the production function in the theory of the firm, yet beyond the existence of equilibrium few general results are known. This paper seeks to remedy that gap.

Keywords: contests, auctions, discontinuous games, all-pay auction, Tullock function

[1]We would like to thank Salvatore Modica and three anonymous referees. We are grateful to the EUI Research Council and MIUR PRIN 20103S5RN3 for financial support.
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1. Introduction

Two themes of Nicholas Yannelis's scientific work are the importance of fundamental results of practical importance and the insistence that they not depend upon special or arbitrary assumptions. So, for example, his work on the existence of competitive equilibrium with large commodity spaces in Yannelis and Zame (1986) does not rest upon arbitrary assumptions about preferences, but it does include the commodity spaces which are important to economists. This paper is about political economy rather than competitive equilibrium, but the analysis and results are in the spirit of Nicholas Yannelis.

Models of two contestants exerting effort to win a prize are common - and of particular importance in the political economy of conflict, such as voting or lobbying. A key element of the analysis is the contest success function giving the probability of winning as a function of the effort of the contestants. This function plays as fundamental a role in the theory of contests as does the production function in the theory of the firm, yet beyond the existence of equilibrium few general results are known. This paper seeks to remedy that gap.

Assumptions about the contest success function vary. In the all-pay auction the greatest effort wins the prize. The widely used Tullock function supposes that the chance of winning is proportional to effort. A great deal is known about the unique mixed strategy equilibrium in the all-pay auction and a great deal is known about pure strategy equilibria when they exist.[4] However, equilibrium generally involves mixed strategies and except in the case of the all-pay auction and some special cases such as the Tullock function with linear costs very little is known about the structure of mixed strategy equilibria. Here we address the basic question of when it is that lower cost of effort results in greater success - that is a greater probability of winning or a greater payoff - across equilibria and contest success functions.

In the spirit of Nicholas Yannelis we do not assume particular functional forms. Rather, we allow general contest success functions of the type that are important to economists including the possibility that there is a discontinuous probability of winning when there is a tie, and we allow for general continuous cost functions. Nash equilibria always exist: this follows from a fundamental result that Nicholas Yannelis developed together with Pavlo Prokopovych in Prokopovych and Yannelis (2014). We take as our measure of success of a contestant her equilibrium utility as a fraction of the prize - that is, how close the contestant is to achieving the goal of winning the prize at no cost.

We observe first that when the contest success function is continuous and costs are high enough, there will be a unique equilibrium in which neither contestant chooses to provide any effort so that lower cost does not provide greater success. More generally, we should be concerned that it might be the case - as it is in alternative models such as the war of attrition - that there can be preemptive equilibria in which the higher cost contestant provides a high effort and

[4] See the survey of Corchón (2007).

by doing so discourages the lower cost contestant. Then, we prove four main results. First, there cannot be a pre-emptive equilibrium in which the higher cost contestant has greater success. Second, a contestant with a sufficiently great cost advantage always has greater success. Third, if the cost advantage is a homogeneous one, then the lower cost contestant always has greater success. Finally, if we retain the generalized convexity and insensitivity property of the generalized all-pay auction but assume a homogeneous cost advantage the contest is payoff equivalent to the generalized all-pay auction.

We further study the robustness of equilibrium by proving a basic upper hemi-continuity result. The underlying mathematics derives from the study of the convergence of monotone functions on rectangles. This enables us to conclude, for example, that contest success functions that converge pointwise to the all-pay auction do not have pure strategy equilibria. More broadly it shows that greater success is a robust property shared by neighboring contest success functions.

A fundamental result of Whitney (1934) enables us to approximate discontinuous contest success functions by real analytic contest success functions. This is important because most functional forms used by economists are real analytic. Remarkably, considering that little is known in general about the structure of mixed strategy equilibria in games with a continuum of actions, we establish that when the contest success function is real analytic, the support of mixed strategy equilibria must be finite. Hence, for example, if the contest success function is the normal cumulative distribution applied to the difference in effort levels, and the variance decreases to zero so that the contest success function approaches the all-pay auction, then equilibria have finite support converging weakly in the limit to the continuous uniform distribution that is the unique equilibrium of the all-pay auction.

2. The Model

Two contestants $j \in \{1, -1\}$ compete for a prize worth $V_j > 0$ to contestant j. Each contestant chooses an effort level $e_j \geq 0$. The probability of contestant j winning the prize is given by a *contest success function* $0 \leq p(e_j, e_{-j}) \leq 1$ that is symmetric in the sense that it depends on the efforts of the two contestants and not on their names.

The contest success function is assumed to be continuous for $e_j \neq e_{-j}$, non-decreasing in e_j, and it must satisfy the adding-up condition $p(e_j, e_{-j}) + p(e_{-j}, e_j) = 1$. Note that we allow for a discontinuous upward jump in the winning probability when we move away from $e_j = e_{-j}$, but require that when there is a tie the probability of winning is $1/2$. Two standard contest resolution functions have this type of discontinuity: the *all-pay auction* in which the highest effort wins for sure and the *Tullock function* where the probability of winning is given by $e_j^\beta/(e_j^\beta + e_{-j}^\beta)$ with $\beta > 0$ which is discontinuous when there is a tie at zero.

The cost of effort e_j is $V_j c_j(e_j)$ and it is incurred regardless of the outcome of the contest. The function $c_j(\cdot)$ measures cost relative to the value of the prize $V_j > 0$. We assume that $c_j(\cdot)$ is continuous, non-decreasing, it satisfies $c_j(0) = 0$, and for some w_j called the *willingness to bid* $c_j(w_j) = 1$ and if $e_j > w_j$ then $c_j(e_j) > 1$. To avoid degeneracy we assume that for contestant -1 the cost function $c_{-1}(\cdot)$ is strictly increasing at the origin.

The objective function of contestant j is given by $V_j p(e_j, e_{-j}) - V_j c(e_j)$. Since choosing effort higher than the willingness to bid is strictly dominated by choosing zero effort, we may restrict the choice of effort to $[0, W]$, where $W > \max\{w_j, w_{-j}\}$. Hence, a strategy for contestant j is a cdf F_j on $[0, W]$. Define $p(F_j, F_{-j}) \equiv \int_0^W \int_0^W p(e_j, e_{-j}) dF_j(e_j) dF_{-j}(e_{-j})$ and $c_j(F_j) \equiv \int_0^W c_j(e_j) dF_j(e_j)$. A *Nash equilibrium* is a pair of strategies (F_j, F_{-j}) such that for each contestant j and all strategies $\tilde{F}_j$ we have

$$p(F_j, F_{-j}) - c_j(F_j) \geq p(\tilde{F}_j, F_{-j}) - c_j(\tilde{F}_j).$$

Since this is an expected utility model this definition is equivalent to restricting deviations to pure strategies e_j.

Existence of Pure and Mixed Equilibrium

The crucial first step in our analysis is the existence of equilibrium. Our first result establishes that equilibia exist and provides some basic information about what they are like.

Theorem 1. *A Nash equilibrium exists and in every Nash equilibrium the probability of a tie at a point of discontinuity is zero. If both contestants have the same costs there is a symmetric Nash equilibrium. However, if $p(e, e)$ is a point of discontinuity for all $0 \leq e \leq W$ the symmetric equilibrium is not in pure strategies.*

Proof. First, suppose that (W, W) is a point of continuity of $p(b_j, b_{-j})$. Existence follows from two conditions. The first is that the sum of utilities of both contestants is continuous. The second is Monteiro and Page (2007)'s uniform payoff security condition. The latter is satisfied since at a point of discontinuity (e, e) a higher effort $e_1 = e + \epsilon$ implies that the utility (e_1, e_{-1}) cannot be much worse than from (e, e) if e_{-1} is close enough to e. Prokopovych and Yannelis (2014) show that these conditions imply the sufficient conditions of Baye, Tian and Zhou (1993) for the existence of a mixed equilibrium. Second, if (W, W) is a point of discontinuity of $p(b_j, b_{-j})$ we can modify the contest success function in any rectangular neighborhood of (W, W) so that no additional points of discontinuity are introduced and the modified function is a contest success function that is continuous at (W, W). Hence an equilibrium exists in the modified game. If we take the rectanglar neighborhood sufficiently small it is strictly dominated to put positive weight on effort there so the equilibrium strategies of the modified game are also equilibrium strategies of the original game.

Symmetry is not typically studied in the existence literature, but proofs are easily adapted by restricting the best response correspondence to the subset of symmetric best responses. In addition, as it is a simple implication of the tools we develop for studying robustness, we give an alternative proof of existence below in Section 7.

Next we show that if $p(e,e)$ is a point of discontinuity then both contestants cannot have an atom at e so the probability of (e,e) is zero. Notice that this immediately implies that if $p(e,e)$ is a point of discontinuity for all $0 \le e \le W$ a symmetric equilibrium cannot be in pure strategies.

To show that both contestants cannot have an atom at e we show that if F_{-j} has an atom at $e < W$ and $p(e,e)$ is a point of discontinuity then $e_j = e$ is not a best-response by j to F_j. If $e = W$ this is obvious since that effort level is strictly dominated by 0.

Define $p^+(e) = \lim_{\epsilon \to 0^+} p(e+\epsilon, e)$. First we show that if $p(e,e)$ is a point of discontinuity of e then $p^+(e) > 1/2$. Discontinuity implies that there is a sequence $e^n \to (e,e)$ with $\lim p(e^n) \neq p(e)$. From symmetry we may assume $\lim p(e^n) > 1/2$. Fix $e+\epsilon$ where $\epsilon > 0$. For n sufficiently large $e_1^n < e+\epsilon$. Hence $p(e+\epsilon, e_{-1}^n) \ge p(e_1^n, e_{-1}^n)$. Since $p(e+\epsilon, e)$ is a point of continuity of $p(e_1, e_{-1})$ we have $p(e+\epsilon, e) = \lim p(e+\epsilon, e_{-1}^n) \ge \lim p(e_1^n, e_{-1}^n)$. Hence $p^+(e) = \lim_{\epsilon \to 0^+} p(e+\epsilon, e) \ge \lim p(e_1^n, e_{-1}^n) > 1/2$.

The remainder of the proof is to show that when $p^+(e) > 0$ it would be better to choose a little bit more effort than e so as to break the tie and get a jump in the probability of winning at trivial additional cost. Specifically, suppose that $-j$ has an atom $f_{-j}(e)$ at e. If j provides effort $e+\epsilon$ instead of e then j gains at least

$$f_{-j}(e)(p^+(e) - 1/2) + c(e) - c(e+\epsilon).$$

In the limit as $\epsilon \to 0$ this is strictly positive proving the result.[5] $\square$

3. Cost and Success

We are interested in the case in which 1 has a cost advantage. We should emphasize here that as we have normalized by the value of the prize, our notion of cost advantage in all cases is one of relative cost advantage. Our goal is to analyze the extent to which this translates to greater success in the contest. One measure of success is a greater probability of winning: we say that j has *outcome success* if $p(F_j, F_{-j}) > 1/2$ or equivalently $p(F_j, F_{-j}) > p(F_{-j}, F_j)$. This, however, fails to take into account the cost of the resources used in achieving success, so we say that j has *greater success* if $p(F_j, F_{-j}) - c_j(F_j) > p(F_{-j}, F_j) - c_{-j}(F_{-j})$, that is, j gets a greater fraction of achievable utility than $-j$. Notice that while success is defined for arbitrary strategies F_j, F_{-j} it will be of interest only when those strategies are equilibrium strategies.

The simplest notion of cost advantage is that of *a pure cost advantage*: here $e > 0$ results in $c_1(e) < c_{-1}(e)$. We first analyze the *generalized all pay* auction

[5]It may be a bit puzzling when $e = w_j$ to think of contestant j deviating to $e+\epsilon$. Clearly this cannot be optimal. However, the argument shows that although such a deviation to a strictly dominated strategy is suboptimal it does better than e, which is just another way of saying e was not a terribly good idea in the first place.

where $e_j > e_{-j}$ results in $p(e_j, e_{-j}) = q > 1/2$. Here higher effort guarantees a greater chance of winning, but the loser also has a chance of winning: for example this could model an electoral process where there is a chance of corruption. The following result generalizes a well known result[6] for the standard all pay auction for which $q = 1$.

Theorem 2. *In the generalized all-pay auction if* 1 *has a pure cost advantage then in any equilibrium* 1 *has greater success.*

Proof. Define $\overline{e}_{-1} \equiv \max \operatorname{supp} F_{-1}$. Consider the strategy for 1 of providing effort $e_\epsilon \equiv \overline{e}_{-1} + \epsilon$. In the all-pay auction this guarantees a win, so

$$p(F_1, F_{-1}) - c_1(F_1) \geq q - c_1(e_\epsilon).$$

By the continuity of c_1 this implies

$$p(F_1, F_{-1}) - c_1(F_1) \geq q - c_1(\overline{e}_{-1}).$$

Because 1 has a pure cost advantage, the right hand side of the inequality is strictly larger than $q - c_{-1}(\overline{e}_{-1})$. Because $\overline{e}(F_{-1}) \in \operatorname{supp} F_{-1}$ there is a sequence $e^n \to \overline{e}_{-1}$ with

$$p(e^n, F_1) - c_{-1}(e^n) = p(F_{-1}, F_1) - c_{-1}(F_{-1}).$$

By the continuity of c_{-1} this implies

$$q - c_{-1}(\overline{e}_{-1}) \geq p(F_{-1}, F_1) - c_{-1}(F_{-1}).$$

Hence it is indeed the case that 1 has greater success. □

Our goal is to understand how this result extends to more general contest success functions. First of all, however, we want to rule out uninteresting cases where the result of Theorem 2 trivially does not extend.

4. Peaceful Equilibria

Consider the following example.

Example 1. Suppose that $c_1(e_1) = e_1, c_{-1}(e_{-1}) = 2e_{-1}$ so that 1 has a pure cost advantage but that $p(e_j, e_{-j}) \equiv 1/2$ so that effort does not matter. Then the unique equilibrium is for each to provide zero effort so both get 1/2 and neither is more successful.

We define *peaceful* equilibria those in which both contestants choose to incur zero cost of effort and have a probability of winning of 1/2 and, recalling our utility normalization, utility equal to 1/2. In particular neither contestant has greater success or greater outcome success regardless of any cost advantages.

[6] See, for example,Siegel (2014).

To have a *contested* equilibrium in which this is not the case we must rule out situations such as Example 1 in which the cost function rises too fast relative to the steepness of the contest success function. We begin with the relevant definitions.

We start with the possibility that contestant j finds it strictly dominant to provide zero effort, that is, $p(0, e_{-j}) - c_j(0) > p(e_j, e_{-j}) - c_j(e_j)$ for all $e_j > 0$ and all e_{-j}. Since $c_j(0) = 0$ we can rewrite this as $c_j(e_j) > \sup_{e_{-j}} p(e_j, e_{-j}) - p(0, e_{-j})$. This separates the cost from the contest success function, and the right hand side is the same for both contestants. If $p(e_j, e_{-j})$ is continuous at 0 with respect to e_j for every e_{-j} then all sufficiently large $c_j(e_j)$ will satisfy this condition, so we call the condition *very high cost*.

A second possibility is that the strict best response to zero effort is zero effort, that is, $p(0, 0) - c_j(0) > p(e_j, 0) - c_j(e_j)$ for all $e_j > 0$. This we can rewrite as $c_j(e_j) > p(e_j, 0) - p(0, 0)$. Again the right hand side is the same for both contestants, and if $p(e_j, 0) - p(0, 0)$ is continuous at 0 all sufficiently large $c_j(e_j)$ will satisfy this condition. Since $\sup_{e_{-j}} p(e_j, e_{-j}) - p(0, e_{-j}) \geq p(e_j, 0) - p(0, 0)$ very high cost implies this condition, so we call it *high cost*. Notice that when p is discontinuous at $(0, 0)$ as it is in the all-pay auction or the Tullock case, high cost (and by implication very high cost) is ruled out because $c_j(e_j)$ is continuous and $c_j(0) = 0$.

By contrast, we say that contestant j has *low cost* if for some e_j we have $c_j(e_j) < p(e_j, 0) - p(0, 0)$, and in particular high cost and low cost are mutually exclusive.

Theorem 3. *If* 1 *has high cost and* -1 *has very high cost then the unique equilibrium is peaceful and neither provides effort. If both have high cost there is a peaceful equilibrium in which neither provides effort. If* 1 *has low cost all equilibria are contested.*

Proof. Since -1 having very high cost -1 provides zero effort because this is strictly dominant, while 1 having high cost means the unique best response by 1 is also to provide zero effort, so the equilibrium is unique and peaceful. Similarly if both have high cost then each finds it optimal to provide zero effort when the other is doing so. Finally, at a peaceful equilibrium since $c_{-1}(e_{-1})$ is assumed to be strictly increasing at the origin, as we noted above, it must be that -1 provides zero effort. The condition for 1 having low cost may be written as $p(e_1, 0) - c_1(e_1) > p(0, 0) - c_1(0)$ implying that 1 gets strictly more than $1/2$ in equilibrium. This requires that the chance of 1 winning is greater than $1/2$ contradicting the definition of a peaceful equilibrium. □

Note that while this result establishes necessary conditions for a contested equilibrium and sufficient conditions, there is a gap between the two conditions.

5. Contested Equilibria

We now focus on contested equilibria. We first show that even in this case pure cost advantage is not in general sufficient for the cost advantaged contestant to have greater success.

Example 2. Here we construct a contested pure strategy equilibrium in which 1 has a pure cost advantage but -1 has greater success. Take $p(e_j, e_{-j}) = (1/2)+(1/2)(e_j - e_{-j})$ truncated by 0 below and 1 above. The cost function for 1 is $c_1(e_1) = (4/7)(e_1 - 1)$ for $e_1 \geq 1$ and 0 otherwise. For -1 it is $c_{-1}(e_{-1}) = (3/7)e_{-1}$ for $0 \leq e_{-1} \leq 2$ and $6/7+(4/7)(e_{-1}-2)$ otherwise. At $e = 0$ we have $c_1 = c_{-1} = 0$. At $e = 1$ we have $c_1 = 0$, and $c_{-1} = 3/7$. At $e = 2$ we have $c_1 = 4/7$, and $c_{-1} = 6/7$. Above 2 the cost difference remain equal to $2/7$ in favor of -1. So 1 has a pure cost advantage. We claim that $(e_1, e_{-1}) = (1, 2)$ is a pure strategy equilibrium. Here 1 loses for certain and has no cost so gets 0 while -1 wins for sure and has a cost of $6/7$ so gets $1/7$. Hence certainly -1 is more successful. To see this is an equilibrium observe that 1 is indifferent to reducing effort below 1: there is no cost and no chance of winning there. Increasing effort above 1 increases the chances of winning at the rate of $1/2$ while it increases costs at the rate of $4/7$ so in fact $e_1 = 1$ is optimal for contestant 1. For -1 reducing effort below 2 reduces the chances of winning at the rate of $1/2$ but decreases costs only at the rate of $3/7$. Increasing effort above 1 has no effect on the chances of winning but simply increases costs. Hence $e_{-1} = 2$ is optimal for contestant -1.

We introduce two strengthened notions of cost advantage

1. 1 has a *marginal cost advantage* if for $e_2 > e_1$we have $c_1(e_2) - c_1(e_1) < c_{-1}(e_2) - c_{-1}(e_1)$
2. 1 has a *homogeneous cost advantage* if $c_1(e) = \alpha c_{-1}(e)$ for some $0 < \alpha < 1$

Given these notions, we have that homogeneous cost advantage implies marginal cost advantage, and marginal cost advantage implies pure cost advantage. An important special case of homogeneous cost advantage occurs when both contestants have the same absolute cost: for all e we have $V_1 c_1(e) = V_{-1} c_{-1}(e)$. In this case 1 has a homogeneous cost advantage if and only if the prize is valued more highly: $V_1 > V_{-1}$.

The notions of pure, marginal, and homogeneous cost advantage are defined independent of the contest success function. An alternative approach is to relate the size of the cost advantage to measures of the steepness of the contest success function. When the contest success function is very steep, as in the all pay auction, intuitively we expect that very little cost advantage is needed.

A simple but quite strong form of cost advantage is the following: we say that 1 has a *strong cost advantage over* -1 if for some $e_1 > w_{-1}$, where w_j is the willingness to bid defined earlier, we have $c_1(e_1) < p(e_1, w_{-1}) - 1/2$. To understand this condition better fix w_{-1}, -1's willingness to bid. If contest success has a strict increase above this point, a sufficiently low cost for 1 will always have a strong cost advantage. On the other hand, strong cost advantage in the all pay auction requires that $c_1(w_{-1}) < 1/2$, while we know that greater success requires only that $c_1(w_{-1}) < 1$.

For this reason we introduce a weaker condition applied over a broader range of effort levels. We say that 1 has a *uniform cost advantage* over -1 if for any $0 \leq e_{-1} \leq w_{-1}$ there is an $e_1 > e_{-1}$with $c_1(e_1) < c_{-1}(e_{-1}) - (p(e_{-1}, 0) - p(e_1, e_{-1}))$.

Notice that this condition is satisfied in the all-pay auction provided that 1 has a cost advantage. It is also satisfied in a difference model in which $p(e_1, e_{-1}) = p(e_1 - e_{-1}, 0)$ if $c_1(2e_1) < c_{-1}(e_1)$. One particularly important case of a uniform cost advantage arises when there is a common underlying strictly increasing cost function $c_2(e)$ but contestant 1 has an effort advantage of $\overline{e}_1 > 0$, meaning that the probability that 1 wins with underlying effort $\tilde{e}_1$ is given by $p(\tilde{e}_1 + \overline{e}_1, e_{-1})$. This is known in the literature as a "head start advantage." This can be made equivalent to the original model by defining $c_1(e_1) = \tilde{c}_2(e_1 - \overline{e}_1)$ for $e_1 \geq \overline{e}_1$ and 0 otherwise. Notice that in this case the cost advantage cannot be homogeneous.

Finally, we generalize the notion of pure strategy equilibrium. We say that F_1, F_{-1} is a *preemptive equilibrium* if either one distribution first order stochastically dominates the other or the two are equal. Equipped with these new definitions we can state our first main result:

Theorem 4. *In a contested equilibrium 1 has greater success in equilibrium if any of the following conditions are satisfied*

(0) she has a pure cost advantage and -1 does not have outcome success,
(i) she has a marginal cost advantage and the equilibrium is preemptive,
(ii) she has a homogeneous cost advantage,
(iii) she has a strong cost advantage,
(iv) she has a uniform cost advantage.

Notice that in Example 2 while 1 had a pure cost advantage in the range $[1, 2]$, 1 also had higher marginal cost than -1. This possibility is ruled out by marginal cost advantage. With this assumption 1 has greater success in all preemptive equilibria. For pure strategies this trivially "works" since all pure strategy equilibria are preemptive. Unfortunately pure strategy equilibria do not always exist and we do not have general results about when equilibria are preemptive. If we further strengthen the cost advantage assumption to homogeneous cost advantage then we get a general result for all equilibria pure or mixed.

Proof. Suppose that F_1, F_{-1} are an equilibrium. From optimality of F_j and symmetry we have

$$p(F_j, F_{-j}) - c_j(F_j) \geq p(F_{-j}, F_{-j}) - c_j(F_{-j}) = 1/2 - c_j(F_{-j}). \qquad (1)$$

By subtraction we also have

$$p(F_j, F_{-j}) - 1/2 \geq c_j(F_j) - c_j(F_{-j}). \qquad (2)$$

First, we show (0). Suppose that 1 has a pure cost advantage but does not have greater success. Then

$$p(F_{-1}, F_1) - c_{-1}(F_{-1}) \geq p(F_1, F_{-1}) - c_1(F_1) \geq 1/2 - c_1(F_{-1}). \qquad (3)$$

Where the first inequality follows from the fact that 1 does not have greater success, and the second from equation 1. Suppose first-1 is not providing effort.

Then $c_{-1}(F_{-1}) = c_1(F_{-1}) = 0$ so 3 implies $p(F_{-1}, F_1) \geq 1/2$. Moreover p non-decreasing implies $p(F_{-1}, F_1) = p(0, F_1) \leq 1/2$ so $p(F_{-1}, F_1) = 1/2$. Since this is not a peaceful equilibrium it must be that $c_1(F_1) > 0$ so $p(F_1, F_{-1}) - c_1(F_1) = 1/2 - c_1(F_1) < 1/2$ while choosing $e_1 = 0$ gives a utility of $1/2$ contradicting the fact that 1 is playing optimally. Suppose second that -1 is providing effort. By the pure cost advantage equation

$$1/2 - c_1(F_{-1}) > 1/2 - c_{-1}(F_{-1})$$

From equation 3 this gives $p(F_{-1}, F_1) > 1/2$. Consequently -1 has outcome success. This proves (0).

To show (i), notice that from equation 2 with $j = 1$ we have

$$p(F_1, F_{-1}) - 1/2 \geq c_1(F_1) - c_1(F_{-1}).$$

From symmetry this gives

$$-p(F_{-1}, F_1) + 1/2 \geq c_1(F_1) - c_1(F_{-1})$$

or

$$p(F_{-1}, F_1) - 1/2 \leq c_1(F_{-1}) - c_1(F_1). \tag{4}$$

From equation 2 with $j = -1$ we have

$$p(F_{-1}, F_1) - 1/2 \geq c_{-1}(F_{-1}) - c_{-1}(F_1)$$

Hence

$$c_1(F_{-1}) - c_1(F_1) \geq c_{-1}(F_{-1}) - c_{-1}(F_1). \tag{5}$$

Suppose that 1 has a marginal cost advantage. If F_1 first order stochastically dominates F_{-1} or the two are equal then -1 does not have a outcome advantage so 1 has greater success by (0). Suppose instead that F_{-1} first order stochastically dominates F_1. For $e_2 > e_1$ the condition for marginal cost advantage can be written as $c_{-1}(e_2) - c_1(e_2) > c_{-1}(e_1) - c_1(e_1)$. It follows that $c_{-1}(F_{-1}) - c_1(F_{-1}) > c_{-1}(F_1) - c_1(F_1)$. This contradicts equation 5. This shows (i).

Next, we show (ii). Suppose that 1 has a homogeneous cost advantage. From equation 5

$$c_1(F_{-1}) - c_1(F_1) \geq c_{-1}(F_{-1}) - c_{-1}(F_1) = (1/\alpha)\left(c_1(F_{-1}) - c_1(F_1)\right).$$

Since $\alpha < 1$ it follows that $c_1(F_{-1}) - c_1(F_1) \leq 0$. From equation 4

$$p(F_{-1}, F_1) - 1/2 \leq c_1(F_{-1}) - c_1(F_1) \leq 0$$

so -1 does not have an outcome success. There are two possibilities. First, if 1 does not have an outcome success either, then, it must be that $p(F_{-1}, F_1) = 1/2$ so that also $p(F_1, F_{-1}) = 1/2$. By (0) we may assume that -1 does not provide zero effort with probability one so by cost advantage

$$p(F_1, F_{-1}) - c_1(F_{-1}) > p(F_1, F_{-1}) - c_{-1}(F_{-1}) = p(F_{-1}, F_1) - c_{-1}(F_{-1})$$

and indeed 1 instead has greater success. The second possibility is that 1 does have outcome success. In this case by (0) 1 also has greater success. This proves (ii).

We now show (iii). If 1 has a strong cost advantage then there is a $\hat{e}_1$ with $c_1(\hat{e}_1) < p(\hat{e}_1, w_{-1}) - p(w_{-1}, w_{-1}) = p(\hat{e}_1, w_{-1}) - 1/2$. Hence $p(\hat{e}_1, w_{-1}) - c_1(\hat{e}_1) > 1/2$. Observe that $F_{-1} \leq w_{-1}$ so $p(\hat{e}_1, w_{-1}) \leq p(\hat{e}_1, F_{-1})$. Finally, from optimality

$$p(F_1, F_{-1}) - c_1(F_1) \geq p(\hat{e}_1, F_{-1}) - c_1(\hat{e}_1) \geq p(\hat{e}_1, w_{-1}) - c_1(\hat{e}_1) > 1/2$$

which as both contestants cannot have a utility greater than 1/2 implies greater success. This proves (iii)

Finally we prove (iv). Let $\hat{e}_{-1}$ be the top of the support of the equilibrium F_{-1}. Let $e^n_{-1} \leq \hat{e}_{-1}$ with $e^n_{-1} \to \hat{e}_{-1}$ and $p(e^n_{-1}, F_1) - c_{-1}(e^n_{-1}) = p(F_{-1}, F_1) - c_{-1}(F_{-1})$. Since at points of discontinuity of p the jump is up this implies

$$p(F_{-1}, F_1) - c_{-1}(F_{-1}) \leq p(\hat{e}_{-1}, 0) - c_{-1}(\hat{e}_{-1}).$$

From the definition of a uniform cost advantage there is a $\hat{e}_1$ such that

$$p(F_{-1}, F_1) - c_{-1}(F_{-1}) < p(\hat{e}_1, \hat{e}_{-1}) - c_1(\hat{e}_1).$$

Moreover because $\hat{e}_1$ is the top of the support of F_{-1} we get

$$p(F_{-1}, F_1) - c_{-1}(F_{-1}) < p(\hat{e}_1, F_{-1}) - c_1(\hat{e}_1)$$

By optimality of F_1 this gives

$$p(F_{-1}, F_1) - c_{-1}(F_{-1}) < p(F_1, F_{-1}) - c_1(F_1)$$

that is to say, greater success. □

The following special case of parts (i) and (ii) of Theorem 4 is useful in a variety of applications.

Corollary 1. *In a contested equilibrium* 1 *has greater success if either of the following two conditions is satisfied:*

(i) Cost is linear for both contestants and 1 *has a pure cost advantage.*[7]

(ii) 1 *has a marginal cost advantage and one contestant provides no effort.*

[7]This assumption is very popular in the literature.

6. Convexity and An All-Pay Auction Result

Hirshleifer (1989) points out that it is likely to be the case in practice that effort makes the greatest difference when the contest is close. If this is the case, we would expect that the contest success function $p(e_j, e_{-j})$ should be convex in e_j for $e_j < e_{-j}$. He argues that in this case one contestant should be expected to provide zero effort. We now examine this possibility more closely.

Consider a real valued function $h(e)$ on $[0, \infty)$. If the function is continuously differentiable then strict convexity implies that if $h(e) \geq h(0)$ for $e > 0$ then $h'(e) > 0$. We generalize this idea by calling $h(e)$ *generalized convex* up to $\overline{e}$ if for $e \in (0, \overline{e}]$ and $h(e) \geq h(0)$

$$h^{+}(e) \equiv \lim \sup_{\epsilon \to 0^{+}} \frac{h(e+\epsilon) - h(e)}{\epsilon} > 0.$$

Generalized convex functions cannot achieve a maximum in $(0, \overline{e}]$ since $h(e) \geq h(0)$ implies $h^{+}(e) > 0$.

A contest is *generalized convex* if for each contestant j and all $e_{-j} > 0$ the objective $p(e_j, e_{-j}) - c_j(e_j)$ is generalized convex as a function of e_j up to e_{-j}. If cost is strictly positive for $e_1 > 0$ the all-pay auction is generalized convex: the condition $p(e_j, e_{-j}) - c_j(e_j) \geq p(0, e_{-j})$ is never satisfied for $0 < e_j < e_{-j}$,while at $e_j = e_{-j}$ the right derivative is positive infinity.

Hirshleifer (1989)'s argument suggests that contest success functions should be generalized convex. This condition is satisfied by many standard contest success functions. Ewerhart (2017) studies continuously differentiable contest success functions and shows that if the elasticity of the odds ratio with respect to own effort is globally larger than 2 then generalized convexity holds. He shows that if $\beta > 2$ this elasticity condition is satisfied by the Tullock function. He shows in that case that it is also satisfied by the *serial* contest success function $p(e_j, e_{-j}) = (1/2)(e_j/e_{-j})^{\beta}$ for $e_j < e_{-j}$ studied by Alcalde and Dahm (2007).[8]

Generalized convexity not only applies to discontinuous contest success functions, it is weaker than the elasticity condition even for continuously differentiable functions. For example, while the serial contest success fails the elasticity condition for $\beta \leq 2$ it is continuously differentiable and for $e_j \leq e_{-j}$ and $\beta > 1$ strictly convex in e_j so it is generalized convex even for $1 < \beta \leq 2$.

If the contest success function is generalized convex and the cost functions are not "too convex," and certainly if they are weakly concave, then the contest will be generalized convex.[9]

Let us say that a contest is *insensitive* if for each contestant j and $e_{-j} > 0$ we have $p(0, e_{-j}) = q < 1/2$. This is a strong condition but is satisfied in cases such as the Tullock and serial cases where $q = 0$.

If F_j, F_{-j} are an equilibrium, we write $\hat{u}_j = p(F_j, F_{-j}) - c_j(F_j)$ for the corresponding normalized utility. We can then generalize the results of Hirshleifer

[8] Feddersen and Sandroni (2006) study $\beta = 1$ with quadratic cost.

[9] For example cost functions are linear in Ewerhart (2017).

(1989), Alcalde and Dahm (2007) and Ewerhart (2017)

Theorem 5. *(i) If a contest is generalized convex then in any equilibrium there is at least one contestant who provides effort with positive probability in every interval containing zero.*

(ii) If in addition the contest is insensitive then in any equilibrium F_j, F_{-j} neither contestant uses a pure strategy and there is a less successful contestant $-j$ who receives $\hat{u}_{-j} = q$ and and more successful contestant j who receives $\hat{u}_j = (1 - F_{-j}(0))q + F_{-j}(0)(1 - q)$.

(iii) If in addition 1 has a homogeneous cost advantage $c_1(e_1) = \alpha c_{-1}(e_1)$ then in any equilibrium 1 is more successful and $F_{-1}(0) = 1 - \alpha$.

Part (iii) says is that if we retain the generalized convexity and insensitivity property of the generalized all-pay auction but assume a homogeneous cost advantage the contest is payoff equivalent to the generalized all-pay auction with the same costs and q.

Proof. Suppose the contest is generalized convex. Define $\underline{e}_j$ to be the lowest point of support in the equilibrium F_j. If $\underline{e}_j > 0$ then for $-j$ the objective $p(e_{-j}, F_j) - c_{-j}(e_{-j})$ is generalized convex up to $\underline{e}_j$ meaning that it is strictly suboptimal for $-j$ to provide effort in $(0, \underline{e}_j]$. This implies that either one of the $\underline{e}_j$'s is zero or both are equal. If both are equal, Theorem 1 and Lemma 7 in the Appendix imply that for one j the function $p(e_j, F_{-j})$ is continuous in e_j at $\underline{e}_j$ hence so is the objective function. Since $\underline{e}_j$ is strictly suboptimal and $p(e_j, F_{-j})$ is continuous there, it follows that there is an $\epsilon > 0$ such that e_j is strictly suboptimal in $[\underline{e}_j, \underline{e}_j + \epsilon]$ contradicting the definition of $\underline{e}_j$.

Suppose next that $\underline{e}_{-j} = 0$, that the contest is insensitive and that $\underline{e}_j > 0$. We will show this is impossible.

Since $p(0, e_{-j})$ is constant for $e_{-j} > 0$ and $-j$ does not provide effort in $(0, \underline{e}_j]$ define the function $v_j(e_j) = p(e_j, F_{-j}) - c_j(e_{-j})$ for $e_j > 0$ and $v_j(0) = \lim_{e_j \to 0} p(e_j, F_{-j}) - c_j(e_j)$. This is generalized convex up to $\underline{e}_j$.

If p is discontinuous at $(\underline{e}_j, \underline{e}_j)$ and $-j$ has an atom there then j does not by Theorem 1. It follows from Lemma 7 that there is an $\epsilon > 0$ such that such that e_j is strictly suboptimal in $[\underline{e}_j, \underline{e}_j + \epsilon]$. Hence v_j is in fact generalized convex up to $\underline{e}_j + \epsilon$, so for $\hat{e}_j \in [\underline{e}_j, \underline{e}_j + \epsilon]$ we have $v_j(\hat{e}_j) < \lim_{e_j \to 0} p(e_j, F_{-j}) - c_j(e_j)$. Hence $\hat{e}_j$ is not optimal. This contradicts the definition of $\underline{e}_j$.

If either p is continuous at $(\underline{e}_j, \underline{e}_j)$ or $-j$ has no atom there, the generalized convexity of v_j up to $\underline{e}_j$ implies that $v_j(\underline{e}_j) < \lim_{e_j \to 0} p(e_j, F_{-j}) - c_j(e_j)$. By Lemma 7 it follows that there is an ϵ' so that for $\hat{e}_j \in [\underline{e}_j, \underline{e}_j + \epsilon']$ we have $v_j(\hat{e}_j) < \lim_{e_j \to 0} p(e_j, F_{-j}) - c_j(e_j)$. Hence again $\hat{e}_j$ is not optimal, again contradicting the definition of $\underline{e}_j$. As all cases have been covered, we conclude that $\underline{e}_j = 0$ for both contestants.

We next derive the equilibrium normalized utility under the insensitivity assumption. Choose a positive sequence $e_j^n, e_{-j}^n \to 0$ such that $p_j(e_j^n, e_{-j}^n) \to p_0$. Since $\underline{e}_j = 0$ for both contestants we can choose a sequence $\tilde{e}_j^n \leq e_j^n$ in the support of F_j in which $p_j(\tilde{e}_j^n, F_{-j}) - c_j(\tilde{e}_j^n) = \hat{u}_j$. Since cost is continuous

$$\hat{u}_j = \liminf p_j(\tilde{e}^n_j, F_{-j}) = \liminf \int_{0<e_{-j}} \min\{p_0, p_j(\tilde{e}^n_j, e_{-j})\} dF_{-j}(e_{-j}) + F_{-j}(0)(1-p_0)$$

$$+ \int_{0<e_{-j}<e^n_{-j}} \left[p_j(\tilde{e}^n_j, e_{-j}) - \min\{p_0, p_j(\tilde{e}^n_j, e_{-j})\}\right] dF_{-j}(e_{-j})$$

$$\leq (1 - F_{-j}(0))q + F_{-j}(0)(1-q).$$

Since at least $(1-F_{-j}(0))q + F_{-j}(0)(1-q)$ is obtained by providing zero effort, it follows that in fact

$$\hat{u}_j = (1 - F_{-j}(0))q + F_{-j}(0)(1-q). \tag{6}$$

Since insensitivity implies discontinuity at zero, by Theorem 1 both contestants do not have an atom at zero. If j has no atom then $-j$ gets q. If $-j$ provides zero effort with probability one then j has no best response so this is not an equilibrium. Since $\underline{e}_j = 0$ it must be that j is mixing as well. This proves (ii).

To prove (iii) observe that if cost is homogeneous it follows from Theorem 4 (ii) that 1 must be more successful. The final part of the argument is derived from Ewerhart (2017) and Alcalde and Dahm (2007). Consider the contest in which 1 has cost $(\alpha/(1-F_{-1}(0)))c_{-1}(e_1)$. Then the strategies F_1 and $F_{-1}/(1-F_{-1}(0))$ on $e_{-1} > 0$ are an equilibrium of this modified game. Moreover both contestants get p_0 as neither has an atom at zero. By Theorem 4 this implies $\alpha/(1-F_{-1}(0)) = 1$. □

7. Robustness and the Equilibrium Correspondence

In order to prove existence we will now deal with sequences of contests $p_n(e_1, e_{-1}), c_{1n}(e_1), c_{-1n}(e_{-1})$. To make sense of this, we now give a slightly more formal definition of a contest. A *contest* on W is a contest success function $p(e_j, e_{-j}) \geq 0$ for $0 \leq e_1, e_{-1} \leq W$, which is non-decreasing in the first argument, continuous except possibly at $e_j = e_{-j}$, and satisfying the adding-up condition $p(e_j, e_{-j}) + p(e_{-j}, e_j) = 1$ together with a pair of cost functions $c_j(e_j) \geq 0$ non-decreasing and continuous with $c_j(0) = 0$, $c_j(W) > 1$, and c_{-1} strictly increasing at 0. For a contest on W we take the strategy space to be of cumulative distribution functions on $[0, W]$. Theorem 13 in the Appendix shows that

Theorem 6. *Suppose p_n, p_0, c_{jn}, c_{j0} are a sequence of contests in W with $p_n(e_1, e_{-1}) \to p_0(e_1, e_{-1}), c_{jn}(e_j) \to c_{j0}(e_j)$ for each $0 \leq e_1, e_{-1} \leq W$ and that F_{1n}, F_{-1n} are equilibria for p_n, c_{jn} converging weakly to F_{10}, F_{-10}. Then $p_n(F_{jn}, F_{-jn}) \to p_0(F_{j0}, F_{-j0})$, $c_{jn}(F_{jn}) \to c_{j0}(F_{j0})$ and F_{10}, F_{-10} is an equilibrium for $p_0(e_1, e_{-1}), c_{j0}(e_j)$.*

We should emphasize that this result requires only the pointwise convergence of p_n, c_{jn}. Pointwise convergence is easy to check, but, as shown in the

Appendix, has strong consequences for non-decreasing functions on rectangles. If the limit is continuous the convergence is uniform. Even if the limit is discontinuous on the diagonal - as we allow for contest success function - convergence is uniform on the set of effort pairs that is bounded away from the diagonal.

As an example the Tullock contest success function converges pointwises to the all-pay auction as $\beta \to \infty$, so in any sequence of equilibria the payoff of -1 converges to zero and that of 1 to $1 - c_1(w_{-1})$. This is a known result. The following implication is new. Say for $b > 0$ that a conflict resolution function is *perturbed* Tullock if $p(e_j, e_{-j}) = (b + e_j)^\beta / \left((e_j + b)^\beta + (e_{-j} + b)^\beta\right)$.[10] Alternatively, a conflict resolution function is *perturbed serial* if $p(e_j, e_{-j}) = (1/2)((e_j + b)/(e_{-j} + b))^\beta$ for $e_j < e_{-j}$. Notice that both of these functions are continuous but fail the insensitivity condition of Theorem 5, never-the-less that theorem together with Theorem 6 imply the following:

Corollary 2. *Suppose that the conflict resolution function is perturbed Tullock with $\beta > 2$ or perturbed serial with with $\beta > 1$ and that 1 has a homogeneous cost advantage $c_1(e_1) = \alpha c_{-1}(e_1)$. Then in the limit as $h \to 0$ in any sequence of equilibria the utility of 1 converges to $1 - \alpha$ and of -1 to zero.*

We say that a contest is *well-behaved* if $p(e_j, e_{-j}) > 0$, p is strictly increasing in the first argument, c_j is strictly increasing, and both have an extension to an open neighborhood of $[0, W] \times [0, W]$ that is real analytic. Some contest success functions studied in the literature have real analytic extensions. This is true of the perturbed Tullock function. The *logit function*

$$p(e_j, e_{-j}) = \frac{\exp(\beta e_j)}{\exp(\beta e_j) + \exp(\beta e_{-j})}$$

introduced by Hirshleifer (1989) is another example. Notice that like the Tullock function as $\beta \to \infty$ the logit function converges pointwise to the all-pay auction. Another example can be found in Shachar and Nalebuff (1999) who take

$$p_j(e_j, e_{-j}) = H\left(\frac{1}{2} + \frac{\exp(e_j) - \exp(e_{-j})}{\exp(e_j) + \exp(e_{-j})}\right)$$

where H is a cdf with support in $[0, 1]$. If the cdf H is symmetric around $1/2$ then $p_j(e_j, e_{-j})$ is a contest success function, and if in addition H admits a real analytic extension to $(-\epsilon, 1 + \epsilon)$ then so does $p_j(e_j, e_{-j})$.

Other contest success functions studied in the literature are not well-behaved either being discontinuous as is the case with the all-pay auction and Tullock function, or having discontinuities in the derivatives as is the case with the *quasi-linear* function $P(e_j - e_{-j})$ which is linear when it is not 0 or 1. Never-the-less in Appendix 12 we show that all contests can be approximated by well behaved contests:

[10] As for example in Amegashi (2006).

Theorem 7. *If p, c_j is a contest on W then there is a sequence of well-behaved contests p_n, c_{jn} on W with $p_n(e_j, e_{-j}) \to p(e_j, e_{-j}), c_{jn}(e_j) \to c_j(e_j)$ for every $(e_1, e_{-1}) \in [0, W] \times [0, W]$.*

Since real analytic functions are continuous, as an immediate corollary we have:

Corollary 3. *Nash equilibria exist. If both contestants have the same costs there is a symmetric Nash equilibrium.*

We are interested in understanding properties of contests that are robust. By a *property* we mean a statement $\Pi(p, c, F)$ such as: there is complete rent dissipation, contestant 1 has greater success, or one contestant has zero utility. We say that a property is true in a contest if it is true for all equilibria of the game. We say that a $\Pi(p, c, F)$ in p, c is *robust* if whenever it is true in p, c then for every sequence p_n, c_n converging pointwise to p, c and for n sufficiently large the property is true in p_n, c_n.

Corollary 4. *Any strict inequality concerning equilibrium utility, probability of winning, or cost is robust.*

Proof. Suppose not. Then there exists a subsequence in which $\Pi(p_n, c_n, F_n)$ is false. Since the space of strategies is compact every subsequence contains a further subsequence that converges weakly to some F. By Theorem 6 F is an equilibrium and utility, winning probability, and cost converge. Hence as the strict inequality is presumed to be satisfied for F for all sufficiently large n it was satisfied for $\Pi(p_n, c_n, F_n)$, a contradiction. $\square$

An important implication of Theorem 6 and Corollary 4 is that if p_n converges to the all-pay auction holding fixed costs c_j then utilities and the probability of winning approach those of the all-pay auction. If we assume that the costs are linear then Ewerhart (2017) shows that if p_n is close enough to the all-pay auction as measured by "decisiveness" then the utility and probability of winning are in fact identical to those of the all-pay auction.

Finite Support

In Appendix 15 we show that well-behaved contests have a relatively simple equilibrium structure:

Theorem 8. *Suppose that $c_1(e_1) = 0$ for $0 \leq e_1 \leq \underline{w}_1$ and if $\underline{w}_1 > 0$ we require that $p(e_j, e_{-j})$ is strictly increasing in the first argument (so in particular in any equilibrium $\lim_{w \to \underline{w}_1^-} F_1(w) = 0$). If $p(e_j, e_{-j}), c_j(e_j)$ have real analytic extensions to an open neighborhood of $[\underline{w}_1, W] \times [0, W]$ then every equilibrium has finite support.*

We note that this result holds also for some contests that are not well-behaved. Che and Gale (2000) show that with quasi-linear contest success and linear costs there is an equilibrium with finite support and they explicitly compute it. Ashworth and Bueno De Mesquita (2009) extend that analysis to the case where one contestant has a head start advantage. Ewerhart (2015) who developed the technique we use in the appendix analyzed the symmetric Tullock contest for large β. That function is not well-behaved since it is discontinuous at zero and without the extension of analyticity below zero the finiteness result fails: with linear costs Ewerhart (2015) shows that the support is countable with a single accumulation point at zero and explicitly computes the equilibrium.

8. Rent Dissipation

An important idea in the literature on contests is that of *complete rent dissipation*, meaning that both contestants get zero, competing so hard that the gains are cancelled by the costs. This is the case in the symmetric all pay auction. Notice that this is ruled out if one contestant provides zero effort, since the other cannot get less than one half as this is obtainable by providing zero effort, and by a contested equilibrium in which one contestant has a greater success.

Example 3. Although complete rent dissipation is often associated with symmetry and the all pay auction, interestingly symmetry, discontinuity, and mixed strategy equilibria are not needed for complete rent dissipation. If $p(b_j, b_{-j})$ is a contest success function with $p(0, b_{-j}) = 0$ and continuous for $(b_j, b_{-j}) \neq 0$, for example the Tullock function, and $\hat{b}_j, \hat{b}_{-j} > 0$ then there are cost functions $c_j(b_j), c_{-j}(b_{-j})$ such that $(\hat{b}_j, \hat{b}_{-j})$ is a pure strategy equilibrium with complete rent dissipation. An example is to take $c_j(b_j) = p_j(b_j, \hat{b}_{-j})$ on $[0, 2\hat{b}_j]$ and $c_j(b_j) = p_j(b_j, \hat{b}_{-j}) + b_j$ for $b_j > 2\hat{b}_j$.

Also important in the literature has been the weaker situation in which one contestant gets nothing - this is the case in every all pay auction, symmetric or not. It turns out that the possibility of a contestant getting nothing is quite exceptional. We say that a property is *generic* if it is robust and if for any p, c_1, c_{-1} for which it is not true there is a sequence p_n, c_{jn} converging pointwise to p, c_j in which it is true.

We formally define properties corresponding to dissipation:

1. *no dissipation*: in equilibrium $c_1(F_1) + c_{-1}(F_{-1}) = 0$
2. *partial dissipation*: in equilibrium $0 < c_1(F_1) + c_{-1}(F_{-1}) < 1$
3. *some dissipation*: in equilibrium $0 < c_1(F_1) + c_{-1}(F_{-1})$
4. *complete dissipation*: in equilibrium $c_1(F_1) + c_{-1}(F_{-1}) = 1$
5. *γ-dissipation*: in equilibrium $c_1(F_1) + c_{-1}(F_{-1}) > \gamma$ where $0 \leq \gamma < 1$

Notice that complete dissipation means γ-dissipation for every $0 \leq \gamma < 1$. Moreover, contested equilibrium implies some dissipation. If in addition one contestant has greater success then there is partial dissipation. Recall that robustness and genericity concern a property that applies to all equilibria. We have

Theorem 9. *Concerning rent dissipation:*

(i) there is a subset of contests with no dissipation that are robust
(ii) the entire set of contests with some (or partial) dissipation is robust
(iii) contests without complete dissipation are generic
(iv) contests with γ-dissipation are robust

Proof. (i) The property of very high cost for j is $c_j(e_j) > \sup_{e_{-j}} p(e_j, e_{-j}) - p(0, e_{-j})$ which is robust by Corollary 4. By Theorem 3 if both contestants have very high cost there is a unique peaceful equilibrium and hence no dissipation.

Part (ii) follows directly from Corollary 4 and the fact that some (partial) dissipation is defined by a strict cost inequality

For (iii) we show the slightly stronger result that both contestants getting positive utility is generic. Strict inequality concerning utility is robust by Corollary 4: this proves that both contestants getting positive utility is robust. We will show that for any p_0, c_{j0} there is a sequence p_n, c_{jn} converging uniformly to p_0, c_{j0} in which each contestant gets positive utility in every equilibrium, and this will complete the proof.

For costs we take $c_{jn} = c_{j0}$. Then take $1 > \lambda_n > 0$ to be a sequence converging to zero and define

$$p_n(e_j, e_{-j}) = (1 - \lambda_n)p_0(e_j, e_{-j}) + \lambda_n \Phi(e_j - e_{-j})$$

where Φ is the standard normal cumulative distribution function. This obviously converges uniformly to $p_0(e_j, e_{-j})$. Moreover, for $0 \leq e_j \leq W$ we have $p_n(e_j, e_{-j}) \geq \lambda_n \Phi(-W)$. Hence providing zero effort gets at least $\lambda_n \Phi(-W) > 0$ so this is obtained by both contestants in any equilibrium.

The proof of (iv) follows from taking an anomalous subsequence and then finding one on which F_n converges. □

Notice that (iii) states that complete dissipation is not robust and (iv) that contests near those with complete dissipation - so for example close to symmetric all pay - have nearly complete dissipation.

9. Extensions

Resource Limits

Some of the existing contest models truncate the effort level above: for example, there might be only a limited number of voters or a budget constraint like in Che and Gale (1996) and Pastine and Pastine (2012). This is ruled out in our model, but as in Levine and Mattozzi (2020) we can approximate the effect by assuming that cost grows rapidly, and in particular becomes greater than the value of the prize, as the limiting effort level is approached. For these approximations our assumptions are satisfied so our results hold.

More generally, a model with a truncated effort level is equivalent to a model in which cost is discontinuous at the truncation point, jumping to a level greater than the value of the prize. Specifically, we now wish to consider the possibility that c_j instead of being continuous on the whole support, it is continuous on $[0, \overline{e}_j]$ where $\overline{e}_j > 0$, $c_j(\overline{e}_j) = \overline{c}_j < 1$, and for $e_j > \overline{e}_j$ we have $c_j(e_j) = c_{Max} > 1$.

Here it is crucial to emphasize that we did not use the continuity of the cost function in proving our results on advantage, so those results extend to this more general class of models. Furthermore, if the contest success function itself is continuous, we show in Theorem 15 that our robustness results continue to hold. This leaves the issue of robustness when both p and c are discontinuous, and here we can go no further. The following example adapted from Levine and Mattozzi (2020) shows that upper hemi-continuity of the equilibrium correspondence can fail in that case.

Example 4. Let the contest success function be that of the all-pay auction, and fix a cost function for both contestants that is linear with constant unit marginal cost up to a resource limit of $\overline{e}_j > 0$. We normalize the value of the prize to be 1, assume that $\overline{e}_j + \overline{e}_{-j} = 1/2$ and let $c_{Max} = 2$. This means that both contestants want to violate their resource limits.

Suppose first that $\overline{e}_1 > \overline{e}_{-1}$. In this case contestant 1 receives a utility of at least $3/4$ and contestant -1 gets a utility of 0. Moreover, it is well-known from the earlier literature that in the unique equilibrium effort is uniform in $(0, \overline{e}_{-1})$ and that -1 has an atom at zero and 1 has an atom at $\overline{e}_{-1}$. The implication of the non-trivial atom at $\overline{e}_{-1}$ means, however, that the tie-breaking rule that each contestant has an equal chance of winning in case of a tie is not consistent with equilibrium. If that is the tie-breaking rule, then -1 should chooses $\overline{e}_{-1}$, guaranteeing at least a 50% chance of winning, and so earning at least $1/2 - 1/4 > 0$ rather than zero as the equilibrium requires. In fact the tie-breaking rule at $\overline{e}_{-1}$ must favor contestant 1 at least to the extent that contestant -1 cannot profit from that effort level. In other words: when both p and c are discontinuous we must allow for endogenous tie-breaking rules.

Second, consider what happens as we pass through the point of symmetry. For $\overline{e}_1 > \overline{e}_{-1}$ contestant 1 earns at least $3/4$ and contestant -1 earns nothing; at $\overline{e}_1 = \overline{e}_{-1}$ both contestants earn $1/4$, while for $\overline{e}_1 < \overline{e}_{-1}$ contestant 1 earns 0 and contestant -1 earns at least $3/4$. In other words: both the individual and aggregate payoff are discontinuous as we pass through the point of symmetry.

Finally, suppose that we approximate the discontinuous cost functions by functions that are linear up to $\overline{e}_j - \epsilon$ then rise steeply to c_{Max} as $\overline{e}_j$ is approached. Levine and Mattozzi (2020) show that in this case as long as $\overline{e}_1 \neq \overline{e}_{-1}$ payoffs are well-behaved in the limit. However, this is not the case when there is symmetry. If $\overline{e}_1 = \overline{e}_{-1}$ then with continuous cost there is complete rent dissipation: both players get zero. However, in the limit both contestants get $1/2 - 1/4$ so we have equilibria with complete rent dissipation converging to one where both contestants get a positive rent.

Asymmetric contest success

We start by considering a broader class of models equivalent to ours. Given the separability of the payoff function, the units of effort do not matter. While it might be natural from an economic point of view to identify effort with number of voters, hours devoted to the cause, or amount of money contributed, the model does not care about the units. Specifically, let $h(e)$ denote a continuous strictly increasing function with $h(0) = 0$, that is, a strictly increasing cost function. We define the contest $p(h(e_j), h(e_{-j})), c_j(h(e_j))$ as *equivalent* to the contest $p(e_j, e_{-j}), c_j(e_j)$. Since $h(e)$ is invertible this is an equivalence relation.

Lemma 1. *Any equilibrium in a contest can be transformed to an equilibrium of any equivalent contest with exactly the same probabilities of winning and costs.*

Proof. If an equilibrium strategy of the h contest is denoted by $F_{hj}(e_j)$ we can map the equilibrium strategies by $F_{hj}(h(e_j)) = F_j(e_j)$ and $F_{hj}(e_j) = F_j(h^{-1}(e_j))$. □

Notice that if for contestant -1 the cost $c_{-1}(e_{-1})$ is strictly increasing, we can take $h(e_{-j}) = c_{-1}^{-1}(e_{-j})$ so that the cost function of -1 is linear and given by $c_{h-1}(e_j) = e_j$. In particlar the statement "cost is convex" has no real meaning: we can change the units of cost so as to make cost concave or convex and get an equivalent contest by suitably modifying the contest success function.

A particular implication of Lemma 1 is that it enable us to study asymmetric contest success functions. Many models assume that one contestant has an advantage in providing effort. Let $h_1(e_1)$ be a strictly increasing continuous function with $h_1(e_1) \geq e_1$ and consider the contest success function $p(h_1(e_1), e_{-1})$ where p satisfies our symmetry assumption. This represents the idea that effort by 1 is "worth more" than effort by -1: for example, in a political contest because 1 has a more appealing platform or more attractive candidate. Using Lemma 1 we can immediately map this back into our framework: the equivalent model is $p(e_1, e_{-1})$ with the cost for 1 given as $c_1(h^{-1}(e_1))$ for $e_1 \geq h(0)$ and 0 otherwise.[11]

Not all asymmetries have this form. In the Shachar and Nalebuff (1999) formulation their model can only be reduced to a standard contest if the cdf H is symmetric around 1/2. In a similar way the model of Coate and Conlin (2004) maps to a standard contest only if the parties are of equal expected size. By contrast Herrera, Levine and Martinelli (2008) allow only effort advantage so their model is equivalent to a standard contest for all parameter values.

External costs

Some models have objective functions of the form $p(e_j, e_{-j}) - c_j(e_j) - b_j(e_{-j})$ where $b_j(e_{-j})$ represents external costs to j of e_{-j} possibily collateral damage inflicted on j by the effort of $-j$. The ethical voter or rule utilitiarian model of Feddersen and Sandroni (2006) has this form, although subsequent authors such as Coate and Conlin (2004) have dropped the exteral effect.

[11]Since it may be that there is an advantage even at zero we need to allow the possibility that c_1 is flat up to $h(0)$.

The point here is that subtracting a function that depends only on the other contestant's choice changes the payoffs of the game but has no effect on equilibrium strategies. In particular, if we agree that it is still sensible to measure achievement of goals by $p(e_j, e_{-j}) - c_j(e_j)$, that is net of collateral damage about which j can do nothing, then $b_j(e_{-j})$ does not matter. It does matter, however, for assessment of the efficiency of different contest success functions: low collateral damage is obviously socially desirable. For example, in choosing between an election and a military conflict both designed to exactly mimic the contest success, the former is preferred because it avoids collateral damage.

10. Conclusion

The goal of this paper has been to establish general results about contests. We characterize cost functions for which there are peaceful and contested equilibria. We then prove four main results. First, a contestant with a sufficiently great cost advantage always has greater success. Second, if the cost advantage is a homogeneous one, then the lower cost contestant always has greater success. Third, if we retain the generalized convexity and insensitivity property of the generalized all-pay auction but assume a homogeneous cost advantage the contest is payoff equivalent to the generalized all-pay auction. Finally, we study the robustness of equilibrium. We prove a basic upper hemi-continuity result and examine approximation by real analytic functions. This enables us to show that properties involving strict inequality are robust and that large classes of examples have equilibria with finite support.

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11. Appendix: Upper Hemi-Continuity

Mathematical Preliminaries

We use the standard order on $\Re^M$ so that $x \geq x'$ means that this is true for each component. Suppose that X is a compact rectangle in $\Re^M$, that $f_n(x), f_0(x)$ are uniformly bounded non-decreasing real valued functions on X Denote by D the set of discontinuities of $f_0(x)$ and by $\overline{D}$ the closure of D.

Lemma 2. *Suppose that $D_o \supset \overline{D}$ is an open subset of X. If for all $x \in X$ we have $f_n(x) \to f_0(x)$ then f_n converges uniformly to f on $X \backslash D_o$.*

Proof. If $X \backslash D_o$ is empty this is true trivially. Otherwise as $X \backslash D_o$ is compact if the theorem fails there is a sequence $x_n \in X \backslash D_o$ with $x_n \to x \in X \backslash D_o$ and $f_n(x_n) \to z \neq f_0(x)$. There are two cases as $z < f_0(x)$ and $z > f_0(x)$. Denote the bottom corner of X as y_0 and the top corner as y_1. Notice that since D_o is open and contains the closure of D, then x has an open neighborhood in which f_0 is continuous.

If $z < f_0(x)$ and $x \neq y_0$ since f_0 is continuous near x there is a $y < x$ with $f_0(y) > z$ and an N such that for $n > N$ we have $x_n > y$. Since f_n is non-decreasing $f_n(x_n) \geq f_n(y)$. Hence $z \geq f_0(y)$ a contradiction. If $x = y_0$ then $f_n(y_0) \to f_0(y_0)$ while $f_n(x_n) \geq f_n(y_0)$. Taking limits on both sides we get $z \geq f_0(y_0)$ a contradiction

If $z > f_0(x)$ and $x \neq y_1$ we have $y > x$ such that $f_0(y) < z$ and an N such that for $n > N$ we have $x_n < y$. This gives $f_n(x_n) \leq f_n(y)$ implying $z \leq f_0(y)$ a contradiction. If $x = y_1$ we have $f_n(x_1) \to f_0(x_1)$ and $f_n(x_n) \leq f_n(x_1)$ and taking limits on both sides we get $z \leq f_0(x_1)$ a contradiction. $\square$

We say that an open set D_o *encompasses* f_0 if there is a closed set $D_1 \subset D_o$ such that the interior of D_1 contains $\overline{D}$. Let $\overline{D}_o$ denote the closure of D_o.

Theorem 10. *Suppose that the probability measures μ_n converge weakly to μ_0. If there is a sequence of sets D_a^m, D_g^m with $D_a^m \cup D_g^m$ encompassing f_0 such that* $\limsup_m \limsup_n \sup_{x \in \overline{D}_a^m} |f_n(x) - f_0(x)| = 0$ *and* $\limsup_m \limsup_n \mu_n(\overline{D}_g^m) = 0$ *then* $\lim \int f_n d\mu_n = \int f_0 d\mu_0$.

Proof. By Urysohn's Lemma there are continuous functions $0 \leq g^m(x) \leq 1$ equal to 1 for $x \in X \backslash D_0^m$ and equal to zero for $x \in D_1^m$. Setting $D_o^m = D_g^m \cup D_a^m$

$$|\int f_n d\mu_n - \int f_0 d\mu_0| \leq |\int g^m f_n d\mu_n - \int g^m f_0 d\mu_0| +$$

$$+ |\int (1 - g^m) f_n d\mu_n - \int (1 - g^m) f_0 d\mu_0| \leq$$

$$\leq |\int g^m f_n d\mu_n - \int g^m f_0 d\mu_0| + |\int_{\overline{D}_0^m} f_n d\mu_n - \int f_0 d\mu_0|.$$

If ϕ_n, ϕ_0 are real numbers and m_n, m_0 are non-negative real numbers we have the inequality $|\phi_n m_n - \phi_0 m_0| \leq |\phi_n - \phi_0|(m_n + m_0)$ so

$$|\int f_n d\mu_n - \int f_0 d\mu_0| \leq |\int g^m f_n d\mu_n - \int g^m f_0 d\mu_0| + \int_{\overline{D}_o^m} |f_n - f_0| d(\mu_n + \mu_0).$$

First we show that $\int_{\overline{D}_o^m} |f_n - f_0| d(\mu_n + \mu_0) \to 0$. Let $\overline{f} = \sup |f_k(x)|$. we have

$$\int_{\overline{D}_o^m} |f_n - f_0| d(\mu_n + \mu_0) \leq \int_{\overline{D}_a^m} |f_n - f_0| d(\mu_n + \mu_0) + \int_{\overline{D}_g^m} |f_n - f_0| d(\mu_n + \mu_0)$$

$$\leq \sup_{x \in \overline{D}_a^m} |f_n(x) - f_0(x)| + \overline{f} \left(\mu_n(\overline{D}_g^m) + \mu_0(\overline{D}_g^m) \right).$$

The first term converges to 0 by hypothesis. For the second, as $\overline{D}_g^m$ is closed and μ_n converges weakly to μ_0 we have $\mu_0(\overline{D}_g^m) \leq \limsup \mu_n(\overline{D}_a^m)$ so

$$\limsup_n \overline{f} \left(\mu_n(\overline{D}_g^m) + \mu_0(\overline{D}_g^m) \right) \leq 2\overline{f} \limsup_n \mu_n(\overline{D}_g^m)$$

giving the first result. Second, write

$$|\int g^m f_n d\mu_n - \int g^m f_0 d\mu_0| \leq |\int g^m |f_n - f_0| d\mu_n + |\int g^m f_0 d\mu_0 - \int g^m f_0 d\mu_n|.$$

Since $g^m f_0$ is continuous by construction we have $\lim_n |\int g^m f_0 d\mu_0 - \int g^m f_0 d\mu_n| = 0$ by weak convergence of μ_n to μ_0.

Finally, we show that $\lim_n |\int g^m |f_n - f_0| d\mu_n = 0$. Denote by D_1^{mo} the interior of D_1^m and $X_1^m = X \backslash D_1^{mo}$. By Lemma 2 $|f_n(x) - f_0(x)| \leq \epsilon_n^m$ for $x \in X_1^m$ where $\lim_n \epsilon_n^m = 0$. As $g^m(x) = 0$ for $x \in D_1^m \supset D_1^{mo}$ we have $g^m |f_n - f_0| \leq \epsilon_n^m$ so that $\int g^m |f_n - f_0| d\mu_n \leq \epsilon_n$. □

Recall that $\overline{D}$ denote the closure of D.

Theorem 11. *Suppose that X is a compact rectangle in $\Re^M$, that $f_n(x), f_0(x)$ are uniformly bounded non-decreasing real valued functions on X, that $f_n(x) \to f_0(x)$ and that the probability measures μ_n converge weakly to μ_0. If $\mu_0(\overline{D}) = 0$ then* $\lim \int f_n d\mu_n = \int f_0 d\mu_0$.

Proof. Take the sets $D_o^m = D_g^m$ to be the open $\epsilon_m \to 0$ neighborhoods of $\overline{D}$ and take $D_a^o = \emptyset$. We may take D_1^m sets to be the closed $\epsilon/2$ neighborhoods of $\overline{D}$: this clearly contains $\overline{D}$ in its interior and is contained in D_o^m. Take D_2^m to be the open $2\epsilon_m$ neighborhoods of D: as these contain $\overline{D}_o^m$ is suffices to show that $\limsup_m \limsup_n \mu_n(D_2^m) = 0$. Since D_2^m is open and μ_n converges weakly to μ we have $\limsup_n \mu_n(D_2^m) \leq \mu_0(D_2^m)$, so we need only prove $\limsup_m \mu_0(D_2^m) = 0$. Since $\cap_m D_2^m = \overline{D}$ we have $\lim_m \mu_0(D_2^m) = \mu_0(\overline{D}) = 0$. □

Upper Hemi-Continuity of the Equilibrium Correspondence

We now consider a *convergence scenario.* Here $p_n(e_1, e_{-1}) \to p_0(e_1, e_{-1})$, $c_{jn}(e_j) \to c_{j0}(e_j)$ is a sequence of contests on W. We take F_{1n}, F_{-1n} to be equilibria for n converging weakly to F_{10}, F_{-10} with μ_{jk} the corresponding measures. We say that the convergence scenario is *upper hemi-continuous* if $p_n(F_{jn}, F_{-jn}) \to p_0(F_{j0}, F_{-j0})$, $c_{jn}(F_{jn}) \to c_{j0}(F_{j0})$ for both j and F_{10}, F_{-10} is an equilibrium for $p_0(e_1, e_{-1}), c_{j0}(e_j)$.

Theorem 12. *If $p_n(F_{jn}, F_{-jN}) \to p_0(F_{j0}, F_{-j0})$ for both j then the convergence scenario is upper hemi-continuous.*

Proof. By Theorem 11 $c_{jn}(F_{jn}) \to c_{j0}(F_{j0})$ on the relevant domain $0 \leq e_j \leq W$. This shows that $u_{jn}(F_{jn}, F_{-jn}) \to u_{j0}(F_{j0}, F_{-j0})$. Next consider j deviating to $e_j \in [0, W]$. Suppose first that e_j is an atom of F_{-j0}. Then this is not a best response. Suppose second that e_j is not an atom of F_{-j0}. Hence the function of e given by $p_0(e_j, e)$ has measure zero with respect to F_{-j0}. If follows from Theorem 11 that $p_n(e_j, F_{-jn}) \to p_0(e_j, F_{-j0})$, so also $u_{jn}(e_j, F_{-jn}) \to u_{j0}(e_j, F_{-j0})$. If $e_j$was a profitable deviation, that is, $u_{j0}(e_j, F_{-j0}) > u_{j0}(F_{j0}, F_{-j0})$, it follows by the standard argument that for sufficiently large n we would have $u_{jn}(e_j, F_{-jn}) > u_{jn}(F_{jn}, F_{-jn})$ contradicting the optimality of F_{jn}. $\square$

In what follows all sequences are of strictly positive numbers.

Lemma 3. *If $\gamma^m \to 0$ then there are sequences $G^n, H^m \to 0$ such that on $[0, W + 2\max\gamma^m]$ we have $\max_{e\in[0,W]} c_{jn}(e + 2\gamma^m) - c_{jn}(e) \leq G^n + H^m$.*

Proof. By Lemma 2 we have c_{jn} converging uniformly to c_{j0} so that

$$\max_{e\in[0,W]} c_{jn}(e + 2\gamma^m) - c_{jn}(e) \leq \max_{e\in[0,W]} c_{j0}(e + 2\gamma^m) - c_{j0}(e) + G_{jn}$$

Since c_{j0} is uniformly continuous on compact intervals $\max_{e\in[0,W]} c_{j0}(e+2\gamma^m) - c_{j0}(e) \leq H_{jm}$. Then take $G^n = \max G_{jn}, H^m = \max H_{jm}$. $\square$

Lemma 4. *Fix sequences $\gamma^m, \theta^m \to 0$. Then there exists a sequence $u^n \to 0$ and $\gamma^m \geq \omega^m$ such that for $0 \leq e_{-j} - e \leq \omega^m$:*

(i) If $p(e + \gamma^m) - 1/2 < \theta^m$ then $\sup_{0\leq e_k - e\leq\omega^m} |p_n(e_j, e_{-j}) - p_0(e_j, e_{-j})| \leq 2\theta^m + u^n$.

(ii) If $p(e + \gamma^m) - 1/2 \geq \theta^m$ then $p_n(e + \gamma^m + \omega^m, e_{-j}) - 1/2 \geq \theta^m/2 - u^n$.

Proof. We may apply Theorem 10 to the functions $p_n(e_j, -x_{-j}), p_0(e_j, -x_{-j})$ on the rectangle $[0, W] \times [-W, 0]$ with $D_o = \{(e_j, x_{-j}) \,||e_j + x_j| < \gamma^m\}$ to conclude that $p_n(e_j, -x_{-j})$ converges uniformly to $p_0(e_j, -x_{-j})$ there. Hence there exists a constant u^m such that for $e_j - e_{-j} \geq \gamma^m$ we have $|p_n(e_j, e_{j-1}) - p_0(e_j, e_{j-1})| \leq u^n$.

Fix e. For (i) Take $\omega^m = \gamma^m$. Take $0 \leq e_k - e \leq \omega^m$. Observe that

$$p_0(e_j, e_{-j}) \leq p_0(e + \omega^m, e) < 1/2 + \theta^m.$$

Since $e + \omega^m - e \geq \gamma^m$ we also have $|p_n(e + \omega^m, e) - p_0(e + \omega^m, e)| \leq u^n$ this implies

$$p_n(e_j, e_{-j}) \leq 1/2 + \theta^m + u^n.$$

Reversing the role of j and $-j$ we see that

$$|p_0(e_j, e_{-j}) - 1/2| < \theta^m,\ |p_n(e_j, e_{-j}) - 1/2| < \theta^m + u^n.$$

Hence $|p_n(e_j, e_{-j}) - p_0(e_j, e_{-j})| < 2\theta^m + u^n$.

For (ii), observe that $p_0(e_j, e_{-j})$ is uniformly continuous on $e_j - e_{-j} \geq \gamma^m$. Hence we may find a $\omega^m > 0$ which without loss of generality we may take to be smaller than γ^m such that for $|e_{-j} - e| \leq \omega^m$ we have $|p_0(e_j, e_{-j}) - p_0(e_j, e)| < \theta^m/2$. Since $p_n(e + \gamma^m + \omega^m, e_{-j})$ is non-increasing in e_{-j} we put this all together:

$$p_n(e+\gamma^m+\omega^m, e_{-j}) \geq p_n(e+\gamma^m+\omega^m, e+\omega^m) \geq p_0(e+\gamma^m+\omega^m, e+\omega^m) - u^n$$

$$\geq p_0(e+\gamma^m+\omega^m, e) - \theta^m/2 - u^n \geq p(e+\gamma^m) - \theta^m/2 - u^n \geq 1/2 + \theta^m/2 - u^n.$$

□

Lemma 5. *For any $\gamma^m \to 0$ there are sequences $G^n, H^m \to 0$ such that for any θ^m and $\omega^m \leq \gamma^m$ and any e with $p_n(e + \gamma^m + \omega^m, e_{-j}) - 1/2 \geq \theta^m/2 - u^n > 0$ for all $0 \leq e_{-j} - e \leq \omega^m$ we have*

$$\min_j \mu_{jn}([e, e + \omega^m]) \leq \frac{G_n + H_m}{\theta^m/2 - u^n}$$

Proof. Given $\gamma^m \to 0$ choose the sequences G^n, H^m by Lemma 3.

Define $m_j \equiv \mu_{jn}([e, e + \omega^m])$. If for one j we have $m_j = 0$ then certainly the inequality holds. Otherwise, consider that if each j plays μ_{jn}/m_j in $[e, e + \omega^m]$ then one of them must have probability no greater than $1/2$ of winning. Say this is j. Consider the strategy for j of switching from μ_{jn} to $\hat{\mu}_{jn}$ by not providing effort in $[e, e + \omega^m]$ and instead providing effort with probability m_j at $e + \gamma^m + \omega^m$. This results in a utility gain of at least

$$m_{-j}\left(\theta^m/2 - u^n\right) - \left(c_{jn}(e + \gamma^m + \omega^m) - c_{jn}(e)\right)$$

$$\geq m_{-j}\left(\theta^m/2 - u^n\right) - \left(c_{jn}(e + 2\gamma^m) - c_{jn}(e)\right) \geq m_{-j}\left(\theta^m/2 - u^n\right) - \left(G^n + H^m\right).$$

As the utility gain cannot be positive, this implies $0 \geq m_{-j}\left(\theta^m/2 - u^n\right) - \left(G^n + H^m\right)$ giving the desired inequality. □

Theorem 13. *Convergence scenarios are upper hemi-continuous.*

Proof. By Theorem 12 it suffices to show $p_n(F_{jn}, F_{-jn}) \to p_0(F_{j0}, F_{-j0})$.

Observe that $p_n(e_j, e_{-j}), p_0(e_j, e_{-j})$ are non-decreasing in the first argument and non-increasing in the second so that the functions on the rectangle $[0, W] \times [-W, 0]$, given by $f_k(x) \equiv p_k(x_j, -x_{-j})$, are uniformly bounded. Define $\mu_n = \mu_{1n} \times \mu_{-1n}$ and $\mu_0 = \mu_{10} \times \mu_{-10}$. From Fubini's Theorem μ_n converges weakly to μ_0. so Theorem 10 applies if we can show how to construct the sets D_a^m, D_g^m.

Fix a sequence $\gamma^m \to 0$. Choose sequences G^n, H^m by Lemma 5 and choose $\theta^m \to 0$ so that $H^m/\theta^m \to 0$. Then choose $u^n \to 0$ and $\omega^m \leq \gamma^m$ by Lemma 4.

We cover the diagonal with open squares of width ω^m. Specifically, for $\ell = 1, 2, \ldots, L$ we take the lower corners κ_ℓ of these squares to be $0, 2\omega^m/3, 4\omega^m/3, \ldots$ until the final square overlaps the top corner at (W, W). There are two types of squares: a-squares where $p(\kappa_\ell + \gamma^m) - 1/2 < \gamma^m$ and g-squares where $p(\kappa_\ell + \gamma^m) - 1/2 \geq \gamma^m$.

We take D_a^m to be the union of the a-squares and D_g^m to be the union of the g-squares. Then for each square ℓ we may take a closed square with the same corner but 3/4rths the width and define D_1 to be the union of these squares. Then $D_o^m = D_a^m \cup D_b^m \supset D_1 \supset \overline{D}$ so that indeed D_o^m encompasses p_0.

Since D_a^m is the union of a-squares, by Lemma 4 (i) we have $\sup_{x \in \overline{D}_a^m} |f_n(x) - f_0(x)| \leq 2\theta^m + u^n$, so indeed $\limsup_m \limsup_n \sup_{x \in \overline{D}_a^m} |f_n(x) - f_0(x)| = 0$ as required by Theorem 10.

For a g-square ℓ we have $0 \leq e_{-j} - e \leq \omega^m$ so by Lemma 4 $p_n(e + \gamma^m + \omega^m, e_{-j}) - 1/2 \geq \theta^m/2 - u^n$. Then by Lemma 5

$$\min_j \mu_{jn}([\kappa_\ell, \kappa_\ell + \omega^m]) \leq \frac{G_n + H_m}{\theta^m/2 - u^n}.$$

We now add up over the g-squares four times, once for the odd numbered ones and once for the even numbered ones. This assures that each sum is over disjoint squares. In each case we first add those for which $j = 1$ has the lowest value of $\mu_{jn}([\kappa_\ell, \kappa_\ell + \omega^m])$ and once for $j = -1$. In each set of indices Λ we get a sum

$$\sum_{\ell \in \Lambda} \mu_{jn}([\kappa_\ell, \kappa_\ell + \omega^m]) \mu_{-jn}([\kappa_\ell, \kappa_\ell + \omega^m]) \leq$$

$$\frac{G_n + H_m}{\theta^m/2 - u^n} \sum_{\ell \in \Lambda} \mu_{-jn}([\kappa_\ell, \kappa_\ell + \omega^m]) \leq \frac{G_n + H_m}{\theta^m/2 - u^n}.$$

This gives a bound

$$\mu_n(\overline{D}_g^m) \leq \frac{G_n + H_m}{\theta^m/2 - u^n}.$$

We then have

$$\limsup_n \mu_n(\overline{D}_g^m) \leq \frac{H_m}{\theta^m/2}$$

and since we constructed the sequences so that $H^m/\theta^m \to 0$ the result now follows from Theorem 10. □

12. Appendix: Smoothing Conflict Resolution Functions

Theorem 14. *If p, c_j is a contest on W then there is a sequence of well-behaved contests p_n, c_{jn} on W with $p_n(e_j, e_{-j}) \to p(e_j, e_{-j}), c_{jn}(e_j) \to c_j(e_j)$ for every $(e_1, e_{-1}) \in [0, W] \times [0, W]$.*

To prove this theorem we first state and prove

Lemma 6. *Suppose that* $p_n(e_j, e_{-j}) \to p_0(e_j, e_{-j})$ *and* $p_{mn}(e_j, e_{-j}) \to_m p_n(e_j, e_{-j})$. *Then there is* $M(n)$ *such that* $p_{M(n)n}(e_j, e_{-j}) \to p_0(e_j, e_{-j})$.

Proof. Define $d(p,q) = \inf\{\gamma | \sup_{|e_j - e_{-j}| \geq \gamma} |p(e_j, e_{-j}) - q(e_j, e_{-j})| \leq \gamma\}$. Then $d(p,q) = 0$ if and only if $p = q$, $d(p,q) = d(q,p)$ and $d(p,q) + d(q,r) \leq 2\max\{d(p,q), d(q,r)\}$. Moreover, $d(p_n, p_0) \to 0$ if and only if $p_n(e_j, e_{-j}) \to p_0(e_j, e_{-j})$. Let $\epsilon_n \to 0$ and take $M(n)$ such that for $m \geq M(n)$ we have $d(p_{mn}, p_n) < \epsilon_n$. Then $d(p_{M(n)n}, p_0) \leq 2\max\{\epsilon_n, d(p_n, p_0)\} \to 0$. □

We now prove Theorem 14.

Proof. By Lemma 6 we can do the perturbations sequentially.

Step 1: Perturb p to get it strictly increasing with strictly positive infimum: take $p_n(e_j, e_{-j}) = (1 - \lambda_n)p(e_j, e_{-j}) + \lambda_n \Phi(e_j - e_{-j})$ where Φ is the standard normal cdf.

Step 2: Given p strictly increasing and positive perturb it to get it strictly increasing, positive and C^2. Let $g_n(x_j|e_j) = (1/W)h_n(x_j/W|e_j)$ where $h_n(\bullet|e_j)$ is the Dirichlet distribution with parameter vector

$$8n^3\Big[(1 - \frac{1/n}{2\sqrt{2}})(e_j/W) + \frac{1/n}{2\sqrt{2}}\frac{1}{2}\Big], 8n^3\Big[(1 - \frac{1/n}{2\sqrt{2}})(1 - e_j/W) + \frac{1/n}{2\sqrt{2}}\frac{1}{2}\Big].$$

This is C^∞ in b_j and $g_n(0|e_j) = g_n(W|e_j) = 0$ and taking $p_n(b_j, b_{-j}) \equiv \int_0^\infty p(x_j, x_{-j})g_n(x_j|b_j)g_n(x_{-j}|b_{-j})dx_j dx_{-j}$ this is certainly strictly positive and C^2. To see that it is strictly increasing observe that increasing b_j increases $g_n(x_j|e_j)$ in first order stochastic dominance. Finally, it is shown in the Web Appendix of Dutta, Levine and Modica (2018) that $\Pr(|x_j - e_j| > 1/n) \leq 1/n$ so that we have pointwise convergence at every continuity point of p. Pointwise convergence on the diagonal is by definition.

Step 3: Given p strictly increasing, positive and C^2 perturb it to get it strictly increasing, positive on $[0,W]\times[0,W]$ and real analytic in an open neighborhood. By Whitney (1934) Theorem 1 we can extend p to be C^1on all of R^2. Take an open neighborhood $\mathcal{W}$ of $[0,W]\times[0,W]$ so that p is strictly positive there. By Whitney (1934) Lemma 5 for each $\epsilon > 0$ we can find a real analytic function $q(b_j, b_{-j})$ with $|q - p| < \epsilon$ and $|Dq - Dp| < \epsilon$ on the closure of $\mathcal{W}$. Then define $Q(b_j, b_{-j}) = q(b_j, b_{-j})/(q(b_j, b_{-j}) + q(b_{-j}, b_j))$.

Remark: The case of c_j is similar but easier. In the final step the real analytic function $q_j(b_j)$ is not necessarily zero at zero so we define $Q_j(b_j) = q_j(b_j) - q_j(0)$. □

13. Appendix: Continuity For Ties

Lemma 7. *Suppose that either* F_{-j} *does not have an atom at* e *or* p *is continuous at* (e,e). *Then* $p(e_j, F_{-j})$ *as a function of* e_j *is right continuous at* $e_j = e$.

Proof. Let $e_j^n \downarrow e$ and write

$$p(e_j^n, F_{-j}) = \int p(e_j^n, e_{-j}) dF_{-j}(e_{-j}) = \int_{|e_{-j}-e|>\epsilon} p(e_j^n, e_{-j}) dF_{-j}(e_{-j}) + \int_{|e_{-j}-e|\leq\epsilon} p(e_j^n, e_{-j}) dF_{-j}(e_{-j}).$$

For the first term from Theorem [monotone-uniform]

$$\int_{|e_{-j}-e|>\epsilon} p(e_j^n, e_{-j}) dF_{-j}(e_{-j}) \to \int_{|e_{-j}-e|>\epsilon} p(e, e_{-j}) dF_{-j}(e_{-j}) \leq \int_{e_{-j}\neq e} p(e, e_{-j}) dF_{-j}(e_{-j}).$$

Hence there is a sequence $\epsilon^n \to 0$ such that

$$\limsup \int_{|e_{-j}-e|>\epsilon^n} p(e_j^n, e_{-j}) dF_{-j}(e_{-j}) \leq \int_{e_{-j}\neq e} p(e, e_{-j}) dF_{-j}(e_{-j}).$$

If F_{-j} does not have an atom at e

$$\int_{|e_{-j}-e|\leq\epsilon^n} p(e_j^n, e_{-j}) dF_{-j}(e_{-j}) \leq \int_{|e_{-j}-e|\leq\epsilon^n} dF_{-j}(e_{-j}) \to 0.$$

Hence $\lim_{n\to\infty} p(e_j^n, F_{-j}) \leq p(e, F_{-j})$.

If p is continuous at (e, e) and letting μ_{-j} be the measure corresponding to F_{-j}

$$\int_{|e_{-j}-e|\leq\epsilon^n} p(e_j^n, e_{-j}) dF_{-j}(e_{-j}) \to p(e, e)\mu_{-j}(e).$$

Hence $\lim_{n\to\infty} p(e_j^n, F_{-j}) \leq \int_{e_{-j}\neq e>\epsilon} p(e, e_{-j}) dF_{-j}(e_{-j}) + p(e, e)\mu_{-j}(e) = p(e, F_{-j})$.

Since by monotonicity $p(e_j^n, F_{-j}) \geq p(e, F_{-j})$ right continuity follows from $\lim_{n\to\infty} p(e_j^n, F_{-j}) \leq p(e, F_{-j})$. □

14. Appendix: Resource Limits

A *resource constrained contest* on W is a contest success function $p(e_j, e_{-j})$ together with a pair of cost functions $c_j(e_j)$ that satisfy the definition of being a contest except that p is required to be continuous and we allow the possibility that c_j instead of being continuous on the entire support is continuous on $[0, \overline{e}_j]$ where $\overline{e}_j > 0$, $c_j(\overline{e}_j) = \overline{c}_j < 1$, and for $e_j > \overline{e}_j$ we have $c_j(e_j) = c_{Max} > 1$. Our goal is to prove:

Theorem 15. *Suppose $p_n(e_1, e_{-1}) \to p_0(e_1, e_{-1})$, $c_{jn}(e_j) \to c_{j0}(e_j)$ for $e_j \neq \overline{e}_{j0}$ are a sequence of resource constrained contests in W, that F_{1n}, F_{-1n} are equilibria for n converging weakly to F_{10}, F_{-10}. Then $p_n(F_{jn}, F_{-jn}) \to p_0(F_{j0}, F_{-j0})$, $c_{jn}(F_{jn}) \to c_{j0}(F_{j0})$ for both j and F_{10}, F_{-10} is an equilibrium for $p_0(e_1, e_{-1})$, $c_{j0}(e_j)$.*

Proof. If c_{j0} is continuous then $c_{jn}(e_j) \to c_{j0}(e_j)$ for all e_j there is nothing new to be proven. We take then the discontinuous case. There are two new things that must be shown. First, we must show that if a deviation to $\overline{e}_{j0}$ against

F_{j0} is profitable then, because we do not have pointwise convergence at $\overline{e}_{j0}$, there is another deviation that is also profitable. Second, we must show that $c_{jn}(F_{jn}) \to c_{j0}(F_{j0})$.

The first is simple: if we take a sequence $e_{jm} \to \overline{e}_{j0}$ strictly from below, the continuity of p_0, c_{j_0} imply that $u_{j0}(e_{jm}, F_{-j}) \to u_{j0}(\overline{e}_{j0}, F_{-j})$ so that for large enough m the deviation $e_{jm} \neq e_{j0}$ is also profitable.

To prove the second we first choose $0 < \epsilon < (c_{Max} - 1)/2$. We observe that for each n (including $n = 0$) the fact that c_{jn} is weakly decreasing and left continuous means that $\{e_j | c_{jn}(e_j) \leq \overline{c}_{j0} + \epsilon\} = [0, e_{jn}(\epsilon)]$ and $\{e_j | c_{jn}(e_j) > \overline{c}_{j0} + \epsilon\} = (e_{jn}(\epsilon), W]$ where it is apparent that $e_{j0}(\epsilon) = \overline{e}_{j0}$. Moreover, we can show that $\lim_n e_{jn}(\epsilon) = \overline{e}_{j0}$. To see that for any $\gamma > \overline{e}_{j0}$ we have $\lim_n c_{jn}(\gamma) = c_{Max}$ implying $\limsup e_{jn}(\epsilon) \leq \gamma$. For any $\gamma < \overline{e}_{j0}$ we have $\lim_n c_{jn}(\gamma) \leq= c_{j0}(\gamma) \leq \overline{c}_{j0}$ implying $\liminf_n e_{jn}(\epsilon) \geq \gamma$.

Second, since p_0 is continuous, pointwise convergence of p_n to p_0 implies uniform convergence and since $\mathcal{W}$ is compact, p_0 is uniformly continuous. It follows that $\Delta(\epsilon) = \inf\{0 \leq e_j^1 - e_j^2 | p_n(e_j^2, e_{-j}) - p_n(e_j^1, e_{-j}) \leq \epsilon\}$ is positive.

Third, we show that for sufficiently large n we have

$$\mu_{jn}((e_{jn}(\epsilon), \overline{e}_{j0} + \Delta(\epsilon/2)/2]) = 0.$$

Suppose that $e_j \in (e_{jn}(\epsilon), \overline{e}_{j0} + \Delta(\epsilon/2)/2])$. Then $c_{jn}(e_j) \geq \overline{c}_{j0} + \epsilon$ while $c_{jn}(\overline{e}_{j0} - \Delta(\epsilon/2)/2]) \leq c_{j0}(\overline{e}_{j0} - \Delta(\epsilon/2)/2]) + \eta_n$ where $\eta_n \to 0$. Since $e_j - (\overline{e}_{j0} - \Delta(\epsilon/2)/2]) \leq \Delta(\epsilon/2)$ it follows that $p_n(e_j, F_{-j}) - p_n(\overline{e}_{j0} - \Delta(\epsilon/2)/2], F_{-j}) \leq \epsilon/2$, while $c_{jn}(e_j) - c_{jn}(\overline{e}_{j0} - \Delta(\epsilon/2)/2]) \geq \epsilon - \eta_n$. Hence for $\eta_n < \epsilon/2$ it is not optimal to play e_j.

Fourth, we show that for sufficiently large n we have $\mu_{jn}((e_{jn}(\epsilon), W]) = 0$. To do so we need only show that for sufficiently large n we have $\mu_{jn}((\overline{e}_{j0} + \Delta(\epsilon/2)/2, W]) = 0$. Since $c_{jn}(\overline{e}_{j0} + \Delta(\epsilon/2)/2) \to c_{Max}$ for all sufficiently large n we have $c_{jn}(\overline{e}_{j0} + \Delta(\epsilon/2)/2) > 1$ and since c_{jn} is non-decreasing $c_{jn}(e_j) > 1$ for all $e_j \geq \overline{e}_{j0} + \Delta(\epsilon/2)/2$. Of course it cannot be optimal to play such an e_j.

Fifth we show that $\mu_{j0}((\overline{e}_{j0}, W]) = 0$. This follows from the fact that it is the countable union of the sets

$$(\overline{e}_{j0} + |e_{jn}(\epsilon) - \overline{e}_{j0}|, W] \subset (e_{jn}(\epsilon), W].$$

Sixth, we construct approximating functions $\tilde{c}_{jn}$. Since $c_{j0}$is continuous on $[0, \overline{e}_{j0}]$ we may choose $\gamma < \overline{e}_{j0}$ so that $c_{j0}(\overline{e}_{j0}) - c_{j0}(\gamma) < \epsilon$. Then for $e_j \leq \gamma$ we take $\tilde{c}_{jn}(e_j) = c_{jn}(e_j)$ and for $e_j > \gamma$ we take $\tilde{c}_{jn}(e_j) = c_{jn}(\gamma)$. Certainly then $\tilde{c}_{jn}$ is non-decreasing and converges pointwise to the non-decreasing function $\tilde{c}_{j0}$. It follows that the convergence is uniform, hence $\tilde{c}_{jn}(F_{jn}) \to \tilde{c}_{0n}(F_{j0})$.

Seventh, we bound

$$|\tilde{c}_{jn}(F_{jn}) - c_{jn}(F_{jn})|$$

$$\leq \int_{[0,\gamma]} |\tilde{c}_{jn}(e_{jn}) - c_{jn}(e_{jn})|\, dF_{jn} + \int_{(\gamma, e_{jn}(\epsilon)]} |\tilde{c}_{jn}(e_{jn}) - c_{jn}(e_{jn})|\, dF_{jn}$$

$$
\begin{aligned}
&+\left|\int_{(e_{jn}(\epsilon),W]} (\tilde{c}_{jn}(e_{jn}) - c_{jn}(e_{jn}))\, dF_{jn}\right| \\
&= \int_{(\gamma,e_{jn}(\epsilon)]} |\tilde{c}_{jn}(e_{jn}) - c_{jn}(e_{jn})|\, dF_{jn} \\
&\leq \sup_{(\gamma,e_{jn}(\epsilon)]} |\tilde{c}_{jn}(e_{jn}) - c_{jn}(e_{jn})| \\
&= c_{jn}(e_{jn}(\epsilon)) - c_{jn}(\gamma) \\
&\leq |c_{jn}(e_{jn}(\epsilon)) - c_{j0}(\overline{e}_{j0})| + |c_{j0}(\overline{e}_{j0}) - c_{j0}(\gamma)| + |c_{j0}(\gamma) - c_{jn}(\gamma)| \\
&\leq 2\epsilon + \eta_n
\end{aligned}
$$

where $\eta_n \to 0$.

Finally, we put this together to see that for all $0 < \epsilon < 1/2$ and sufficiently large n we have

$$|c_{jn}(F_{jn}) - c_{j0}(F_{j0})| \leq |\tilde{c}_{jn}(F_{jn}) - \tilde{c}_{j0}(F_{j0})| + 4\epsilon + 2\eta_n.$$

It follows that $\limsup |c_{jn}(F_{jn}) - c_{j0}(F_{j0})| \leq 4\epsilon$. This proves the result. □

15. Appendix: Finite Support

Theorem 16. *Suppose that $c_1(e_1) = 0$ for $0 \leq e_1 \leq \underline{w}_1$ and if $\underline{w}_1 > 0$ we require that $p(e_j, e_{-j})$ is strictly increasing in the first argument (so in particular in any equilibrium $\mu_1([0, \underline{w}_1)) = 0$). Suppose as well that $c_j(W) > 1$. If $p(e_j, e_{-j}), c_j(e_j)$ have real analytic extensions to an open neighborhood of $[\underline{w}_1, W] \times [0, W]$ then every equilibrium has finite support.*

Proof. Take $\underline{w}_{-1} = 0$ and consider

$$U_j(e_j) \equiv \int_{\underline{w}_j}^{W} p(e_j, e_{-j}) dF_{-j}(e_{-j}) - c_j(e_j).$$

We first show that this is real analytic in an open neighborhood of $[\underline{w}_j, W]$. For c_j this is true by assumption so we show it for the integral

$$P_j(e_j) \equiv \int_{\underline{w}_j}^{W} p(e_j, e_{-j}) dF_{-j}(e_{-j}).$$

In Ewerhart (2015) the extensibility properties of $p(e_j, e_{-j})$ were known. Here we must establish them. Let $\mathcal{W}$ be the open neighborhood of $[\underline{w}_1, W] \times [\underline{w}_{-1}, W]$ in which p is real analytic. Then for each point $e \in \mathcal{W}$ the function p has an infinite power series representation with a positive radius of convergence $r_1, r_{-1}$for e_1, e_{-1} respectively. Hence the extension of p to a function of two complex variables has the same radius of convergence there. Take an open square around e_j in the complex plane small enough to be entirely contained in the circle of radius $\min\{r_1, r_{-1}\}$ and lying inside of $\mathcal{W}$. The product of these squares is an open cover of the compact set $[\underline{w}_1, W] \times [\underline{w}_{-1}, W]$, hence has a finite subcover. Choose the smallest square from this finite set, say with length $2h$. Hence $p(e_j, e_{-j})$ is complex analytic in the domain $((\underline{w}_1 - h, W + h) \times (-h, +h)) \times \big((\underline{w}_{-1} - h, W + h) \times (-h, +h)\big)$.

The remainder of the proof follows Ewerhart (2015) in showing that we may extend $P_j(e_j)$ to a complex analytic function in the domain $(\underline{w}_j - h, W + h) \times (-h, +h)$. As this is a convex domain, take a triangular path Δ in this domain and integrate

$$\oint_{\Delta} P_j(e_j) = \oint_{\Delta} \int_{\underline{w}_j}^{W} p(e_j, e_{-j}) dF_{-j}(e_{-j}).$$

Everything in sight is bounded so we may apply Fubini's Theorem and interchange the order of integration to find

$$\oint_{\Delta} P_j(e_j) = \int_{\underline{w}_j}^{W} \left(\oint_{\Delta} p(e_j, e_{-j}) \right) dF_{-j}(e_{-j}).$$

By Cauchy's Integral Theorem since p is analytic $\oint_{\Delta} p(e_j, e_{-j}) = 0$. Hence $\oint_{\Delta} P_j(e_j) = 0$ so by Morera's Theorem $P_j(e_j)$ is analytic, and in particular real analytic when restricted to $(\underline{w}_j - h, W + h) \times 0$.

Hence the gain from deviating to e_j is given by a real analytic function $U_j(e_j) - \max_j U_j(\tilde{e}_j)$. That implies it is either identically zero or has finitely many zeroes. We can rule out the former case since $\max_j U_j(\tilde{e}_j) \leq 1$ and $c(W) > 1$. Hence F_j must place weight only on the finitely many zeroes. $\square$