Solutions Key to

Problem Sets in Game Theory

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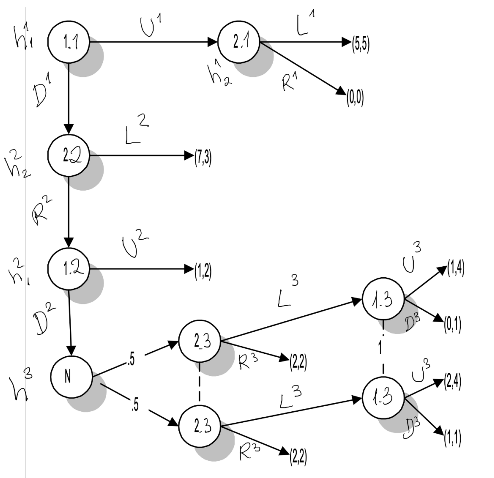
Problem Set 2: Problems on Dyamic Games

Exercise 1. Extensive Form

In the game below find: the normal form, all pure and mixed NE and all SPNE.

Let me label the actions of each player (subscript) according to the information set (superscript) in which they are.

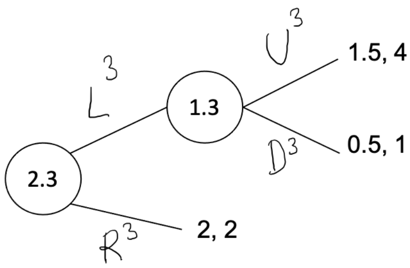


1) *The normal form.*

Let's carefully build the normal form. The first thing to notice is that each player has three information sets (they both share the last information set where nature plays). Since each player has two actions per information set, and there are three information sets (h), each

*This version builds on the solutions provided by Damiano Argan and Konuray Mutluer.

player will have $2^3 = 8$ strategies in the normal form representation. Furthermore, in the strategy profile where nature plays as well, we must calculate the expected utility of each strategy profile.



In normal form:

	L^3	R^3
U^3	1.5, 4	2, 2
D^3	0.5, 1	2, 2

The normal form of the game as a whole:

		P2							
		$R^1R^2R^3$	$R^1R^2L^3$	$R^1L^2R^3$	$R^1L^2L^3$	$L^1R^2R^3$	$L^1R^2L^3$	$L^1L^2R^3$	$L^1L^2L^3$
P1	$U^1U^2U^3$	0 , 0	0 , 0	0 , 0	0 , 0	$\underline{5}$, $\underline{5}$	$\underline{5}$, $\underline{5}$	5 , $\underline{5}$	5 , $\underline{5}$
	$U^1U^2D^3$	0 , 0	0 , 0	0 , 0	0 , 0	$\underline{5}$, $\underline{5}$	$\underline{5}$, $\underline{5}$	5 , $\underline{5}$	5 , $\underline{5}$
	$U^1D^2U^3$	0 , 0	0 , 0	0 , 0	0 , 0	$\underline{5}$, $\underline{5}$	$\underline{5}$, $\underline{5}$	5 , $\underline{5}$	5 , $\underline{5}$
	$U^1D^2D^3$	0 , 0	0 , 0	0 , 0	0 , 0	$\underline{5}$, $\underline{5}$	$\underline{5}$, $\underline{5}$	5 , $\underline{5}$	5 , $\underline{5}$
	$D^1U^2U^3$	1 , 2	1 , 2	$\underline{7}$, $\underline{3}$	$\underline{7}$, $\underline{3}$	1 , 2	1 , 2	$\underline{7}$, $\underline{3}$	$\underline{7}$, $\underline{3}$
	$D^1U^2D^3$	1 , 2	1 , 2	$\underline{7}$, $\underline{3}$	$\underline{7}$, $\underline{3}$	1 , 2	1 , 2	$\underline{7}$, $\underline{3}$	$\underline{7}$, $\underline{3}$
	$D^1D^2U^3$	2 , 2	$\underline{1.5}$, $\underline{4}$	$\underline{7}$, 3	$\underline{7}$, 3	2 , 2	1.5 , $\underline{4}$	$\underline{7}$, 3	$\underline{7}$, 3
	$D^1D^2D^3$	2 , 2	0.5 , $\underline{1}$	$\underline{7}$, $\underline{3}$	$\underline{7}$, $\underline{3}$	2 , 2	0.5 , 1	$\underline{7}$, $\underline{3}$	$\underline{7}$, $\underline{3}$

2) *All pure and mixed Nash equilibria.*

Pure strategies:
The normal form of the game contains the BR of each player underlined. There is a considerable amount of equilibria, so let us use a more concise formulation.

A strategy s_i contains an action for each player i in each information set: $s_i = (a_i^1, a_i^2, a_i^3)$:

$$\begin{aligned} PSNE = \{s_1, s_2\} \in & \left\{\left(U^1 \cdot\cdot\right), \left(L^1R^2\cdot\right)\right\} \\ & \cup \left\{\left(\left(D^1U^2\cdot\right) \cup \left(D^1D^2D^3\right)\right), \left(\cdot L^2\cdot\right)\right\} \\ & \cup \left\{\left(D^1D^2D^3\right), \left(R^1R^2L^3\right)\right\} \end{aligned}$$

where $\cdot$ refers to player i playing either of his two available actions.

Mixed strategies: To simplify the search of mixed strategies, we can write a condensed normal form of the game. Note that I omit strategy $R^1R^2R^3$ for player two as it is strictly dominated by $L^1L^2L^3$. Again, the BRs of each player are underlined. (Note how they coincide with those of the original game). If you develop the indifference conditions, you will find:

		P2				
		$R^1R^2L^3$	$R^1L^2\cdot$	$L^1R^2R^3$	$L^1R^2L^3$	$L^1L^2\cdot$
P1	$U^1\cdot\cdot$	$0, 0$	$0, 0$	$\underline{5}, \underline{5}$	$\underline{5}, \underline{5}$	$5, \underline{5}$
	$D^1U^2\cdot$	$1, 2$	$\underline{7}, \underline{3}$	$1, 2$	$1, 2$	$\underline{7}, \underline{3}$
	$D^1D^2U^3$	$\underline{1.5}, \underline{4}$	$\underline{7}, 3$	$2, 2$	$1.5, \underline{4}$	$\underline{7}, 3$
	$D^1D^2D^3$	$0.5, 1$	$\underline{7}, \underline{3}$	$2, 2$	$0.5, 1$	$\underline{7}, \underline{3}$

$$
\begin{aligned}
MSNE = \Big\{ & \left((U^1\cdot\cdot), \frac{3}{5}(L^1L^2\cdot) + \frac{2}{5}(L^1R^2R^3)\right), \left((U^1\cdot\cdot), \frac{7}{11}(L^1L^2\cdot) + \frac{4}{11}(L^1R^2L^3)\right), \\
& \left((D^1D^2U^3), \frac{3}{10}(L^1R^2L^3) + \frac{7}{10}(R^1R^2L^3)\right), \left(\frac{1}{2}(D^1U^2\cdot) + \frac{1}{2}(D^1D^2U^3), (L^1L^2\cdot)\right), \\
& \left(\frac{1}{2}(D^1U^2\cdot) + \frac{1}{2}(D^1D^2U^3), (R^1L^2\cdot)\right), \left(\frac{2}{3}(D^1D^2U^3) + \frac{1}{3}(D^1D^2D^3), L^1L^2\cdot)\right), \\
& \left(\frac{2}{3}(D^1D^2U^3) + \frac{1}{3}(D^1D^2D^3), (R^1L^2\cdot)\right) \Big\}
\end{aligned}
$$

3) *All subgame perfect equilibra.*

We are looking for a strategy profile that yields a NE in every subgame. Recall the definition of subgame: an information set that begins with a single node. The smallest subgame at the end of the tree in this exercise begins at information set h_1^2. [1] Let me find the NE in this subgame using the normal form. Note that the strategies played by player 1 will have two actions.

	L_3	R_3
U^2U^3	$1, \underline{2}$	$1, \underline{2}$
U^2D^3	$1, \underline{2}$	$1, \underline{2}$
D^2U^3	$\underline{1.5}, \underline{4}$	$\underline{2}, 2$
D^2D^3	$0.5, 1$	$\underline{2}, \underline{2}$

This subgame has two NE:

$$NE = \underbrace{(D^2U^3, L^3)}_{\text{Eq'm A}}, \underbrace{(D^2D^3, R^3)}_{\text{Eq'm B}}$$

This means that any SPNE of the whole game must play one of these two strategy profiles in the last subgame. I have named the two equilibria, so that we look for the BR of the players in previuos stages of the game.

Before making our way up on the tree, let me briefly focus on the subgame beggining at information set h_2^1: when asked to play at this subgame, player 2 will choose L^1, yielding a payoff of $(5, 5)$.

[1] This is crucial, recall our discussion during the TA class. The smallest subgame does not start at information set h^3, because we do not consider nature as a player *per se* and each player has two decision nodes in this information set.

A. Consider equilibrium A, and let's see what other players will respond.

- At h_2^2: $u_2(R^2|\text{eq'm A}) = 4 > 3 = u_2(L^2|\text{eq'm A})$. Player 2 will choose R^2.
- At h_1^1: $u_1(U^1|\text{eq'm A}) = 5 > 1.5 = u_1(D^1|\text{eq'm A})$. Player 1 will choose U^1.

$$SPNE = (U^1 D^2 U^3, L^1 R^2 L^3)$$

B. Consder equilibrium B, and let's see what other players will respond.

- At h_2^2: $u_2(R^2|\text{eq'm B}) = 2 < 3 = u_2(L^2|\text{eq'm B})$. Player 2 will choose L^2.
- At h_1^1: $u_1(U^1|\text{eq'm B}) = 5 < 7 = u_1(D^1|\text{eq'm B})$. Player 1 will choose D^1.

$$SPNE = (D^1 D^2 D^3, L^1 L^2 R^3)$$

Exercise 2. Backward Induction

There are five pirates with names 1,2,3,4,5. They have just seized a hundred gold coins, and now it's time to share the loot. The bargaining rules are: Whoever has the lowest number as a name must propose an division of the one hundred coins to the remaining pirates. If the majority accepts the proposal, then the coins are allocated and the game ends. If the majority does not accept, then the proposer gets thrown overboard and the game is repeated with one less pirate. What should the first pirate propose?

We will assume that in the case of a tie, the pirate gets thrown overboard. Getting thrown overboard yields negative utility, and when indifferent pirates will reject an offer. We will proceed by backward induction. We start at the last subgame of the game.

- Last subgame: There are only P4 and P5 left, as P1, P2, and P3 have been thrown overboard. P4 is the one to make an offer. Whatever he offers, however, he will be thrown to overboard. This is because P5 can always refuse, and in the case of a tie P4 gets thrown and P5 gets all the coins. So refuse is a dominant strategy for P5.

- Second to last subgame: We substitute the last subgame with its equilibrium payoff. P4 knows that if P3's offer gets refused, he will be thrown. P3, at the same time, wants to maximize his payoffs and offer P4 the strict minimum for him to accept. Hence, P3 offers 1 coin to P4, 0 to P5 and keeps 99. With this offer, P3 and P4 vote in favour and P5 against.

$$Off_{P3} = (\cdot, \cdot, 99, 1, 0)$$

- Third to last subgame: There are four pirates left. P2 will anticipate P3's offer. Hence, he understands that he will need to lure two pirates to accept his offer. The *cheapest* ones are P4 and P5. He therefore proposes 1 gold coin to 5, 2 gold coins to 4 and keeps 97 to himself, leaving 0 for P3. His proposal is accepted and this ends the game.

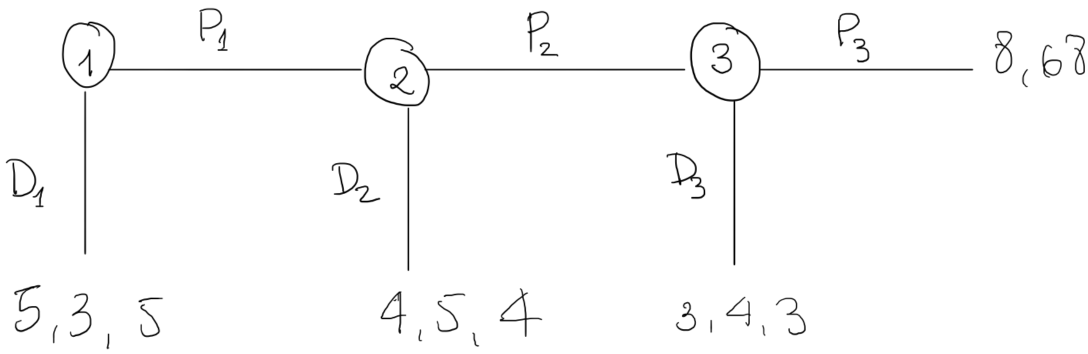
$$Off_{P2} = (\cdot, 97, 0, 2, 1)$$

- First subgame: P1 foresees P2's proposal. He understands that he needs to get two more pirates to accept his proposal. He offers 1 gold coin to P3, 2 coins to P5 and keeps 97 to himself. The pirates will split the money this way and the game will end. The proposal by P1, which eventually will be the accepted one is:

$$Off_{P1} = (97, 0, 1, 0, 2)$$

Exercise 3: Self-Confirming Equilibrium

Consider a three person centipede game in which player 1 can drop or pass, player 2 can drop or pass, and player 3 can drop or pass. If player 1 drops, the payoffs are (5,3,5); if player 2 drops the payoffs are (4,5,4), if player 3 drops the payoffs are (3,4,3) and if player 3 passes the payoffs are (8,6,8).



What payoffs are possible in Nash equilibrium?

To find **all** the NE possible **payoffs** we have to write the normal form representation of this game. There are two games where P2 and P3 play: in the first one P1 has passed, and in the second one P1 has dropped. P2 and P3 will decide what to play , but then P1 will decide which of the two games to play.

Player 1 D1			
	P3		
P2		P_3	D_3
	P_2	5,3,5	5,3,5
	D_2	5,3,5	5,3,5

Player 1 P1			
	P3		
P2		P_3	D_3
	P_2	8,6,8	4,5,4
	D_2	3,4,3	4,5,4

We can hence find the pure strategies NE:

$$PSNE = \{(P_1, P_2, P_3), (D_1, D_2, D_3), (D_1, D_2, P_3), (D_1, P_2, D_3)\}$$

To look for a mixed strategy, let's first assume σ is a strategy profile that is NE and contains $1 > \sigma_1(P_1) > 0$, so that the subgame containing P2's decision is on the equilibrium path. Then P2

<table>
<tr><td rowspan="4">P2</td><td colspan="3">P3</td></tr>
<tr><td></td><td>(p)
P_3</td><td>$(1-p)$
D_3</td></tr>
<tr><td>(q) P_2</td><td>6,8</td><td>4,3</td></tr>
<tr><td>$(1-q)$ D_2</td><td>5,4</td><td>5,4</td></tr>
</table>

and P3 face the game depicted below.

$$u_2(P_2) = 6p + 4(1-p) = 5p + 5(1-p) = u_2(D_2) \qquad p = 1/2$$
$$u_3(P_3) = 8q + 4(1-q) = 3q + 4(1-q) = u_3(D_3) \qquad q = 0$$

There are three NE:
$$\left\{(P_2, P_3), (D_2, D_3), \left(D_2, \frac{1}{2}P_3 + \frac{1}{2}D_3\right)\right\}$$

Let's look at these NE from the perspective of P1: if P2 and P3 play (P_2, P_3), then for P1 playing P_1 with certainty is strictly better off: $8 > 5$; if they play (D_2, D_3) then for P1 playing D_1 is a strictly better: $5 > 4$, and the same is for $(D_2, D_3/2 + P_3/2)$. So, there is no strategy profile where 1 fully mixes. In addition, once player 1 plays P_1, there is no mixed strategy that is Nash. All the mixed strategies that could be Nash have D_1 played with certainty and thus whatever P2 and P3 do, the payoff profile will always be (5,3,5).

Bottom line: the possible payoffs of the NE of this game are are (8,6,8) and (5,3,5).

What payoffs are possible in sequential equilibrium?

To find the sequential rational equilibrium we use backward induction. Start form the last subgame where P3 has to choose between D_3 and P_3. P_3 is a strictly dominant strategy and so is the NE of this subgame. Then replace P3's node by the payoff profile entailed by this choice: (8,6,8). Now, take the subgame starting from P2's decision. P_2 is a strictly dominant strategy for P2 and so it is his choice. Again, we replace the above mentioned subgame by the payoff from P2's choice. Then, for P1 P_1 is a strictly dominant strategy. (P_1, P_2, P_3) is the only sequential rational equilibrium. So, the only possible payoff profile is (8,6,8).

Construct a self-confirming equilibrium that is NOT a public randomization over Nash equilibrium.

The history $\{P_1, D_2\}$ is not part of the set of NE and hence cannot be obtained via public randomization. In other words, there is no way to randomize over NE and have P_1 played with certainty but not P_2. Let's hence consider the strategy profile $\sigma = (P_1, P_2/2 + D_2/2, P_3)$.

In a self-confirming equilibrium, for all the strategies that are played with positive probability, there exist some beliefs that are true on path and maximize players' utility given the beliefs.We therefore need to construct players' beliefs, and recall that we need a belief for each action played with positive probability. We can have two types of equilibria depending on the beliefs we create:

Heterogeneous SCE: create two different beliefs to support the two actions in the support of P2's strategy:

- P1 believes that P3 will pass and that P2 will pass only 50% of the time.

$$\mu_1[\Pr(D_2) = 1/2, \Pr(D_3) = 0] = 1$$
$$u_1(P_1, \sigma_{-i}) = \frac{1}{2}4 + \frac{1}{2}8 = 6 > 5 = u_1(D_1, \sigma_{-i})$$

- P2: we need to construct a belief for each action played with positive probability.
  - μ_{P2}: believes that P3 will pass. Note that the belief needs to be true on the equilibrium path.

$$\mu_{P2}[\Pr(P_1) = 1, \Pr(P_3) = 1] = 1$$
$$u_2(P_2, \sigma_{-i}) = 6 > 5 = u_2(D_2, \sigma_{-i})$$

  - μ_{D2}: for D_2 to be a BR, we want to construct a belief off the equilibrium path that satisfies:

$$\mu_{D2}[\Pr(P_1) = 1, \Pr(P_3) = 0] = 1$$
$$u_2(D_2, \sigma_{-i}) = 5 > 4 = u_2(P_2, D_3)$$

    Note that this belief need not be correct, as what P3 plays is off the equilibrium path and P2 will never realise his mistake.

- P3 will pass. [2]

$$\mu_3[\Pr(P_1) = 1, \Pr(P_2) = 1] = 1$$
$$u_3(P_3, \sigma_{-i}) = 8 > 3 = u_3(D_3, \sigma_{-i})$$

Unitary SCE: alternatively, we could assign the same beliefs to both strategies of player 2:

- P1 believes that P3 will pass and that P2 will pass only 50% of the time.

$$\mu_1[\Pr(D_2) = 1/2, \Pr(D_3) = 0] = 1$$
$$u_1(P_1, \sigma_{-i}) = \frac{1}{2}4 + \frac{1}{2}8 = 6 > 5 = u_1(D_1, \sigma_{-i})$$

- P2: we will assing the same beliefs to both actions of player 2.

$$\mu_2[\Pr(P_1) = 1, \Pr(P_3) = 1/2] = 1$$
$$u_2(P_2, \sigma_{-i}) = \frac{1}{2}(6 + 4) = 5 = u_2(D_2, \sigma_{-i})$$

  P2 is indifferent, he has no incentive to deviate.

[2] We **DO** need to construct beliefs for the player at the last node of the game. This will allow us to eventually generate equilibria that have player 3 mixing or even dropping all together. Note that, as any NE is a SCE with the correct beliefs, and we have seen that in some NE P3 drops, we should indeed be able to construct SCE where P3 drops.

- P3 will pass.

$$\mu_3[\Pr(P_1) = 1, \Pr(P_2) = 1] = 1$$
$$u_3(P_3, \sigma_{-i}) = 8 > 3 = u_3(D_3, \sigma_{-i})$$

Either of these SCE fulfill the conditions: beliefs are correct on path, and agents maximize given the beliefs.

Exercise 4: Chain Store Game

Consider the following chain store game played between a patient player one (chain store) with discount factor δ and a sequence of short-run myopic player 2's (entrants – with discount factor 0)

		Ent (P2)	
Inc (P1)		Out	In
	fight	3,0	-2,-2
	give in	4,0	2,2

a. *What is the Nash equilibrium if the game is played once?*

   For the incumbent, *fight* is strictly dominated by *give in*. Once, you remove *fight*, *out* it is strictly dominated by *in*.

   $$PSNE = \{(G, I)\}$$

b. *What is the Stackelberg equilibrium in which player 1 gets to commit if the game is played once?*

   The incumbent can commit using pure strategies and mixed strategies.

   In pure strategies, if he commits to playing fight, the entrant's BR is to play out and he gets a payoff of 3. If, on the other hand, he commits to playing give in, the entrants BR is to play in and he gets a payoff of 2. He is better off committing to fighting and getting a payoff of 3.

   In Stackelberg with mixed strategies, we will use the following tie-breaking rue: when indifferent, the respondent will take the action that is in the best interest of the other player. [3]

   In this case, P1 gets to commit first. He will be the one forcing an indifference condition for P2's strategies. Call α_F the probability that P1 plays F.

   $$u_2(\alpha_F, O) = 0 = -2\alpha_F + 2(1 - \alpha_F) = u_2(\alpha_F, I) \qquad \alpha_F = 1/2$$

[3] We had a discussion during the TA class about this. The idea is that an equilibrium strategy cannot contain ε, because one could always choose a smaller ε, say $\varepsilon/2$, which would constitute a profitable deviation and hence break the equilibrium.

$$BR_2(\alpha_F) = \begin{cases} O & \text{if } \alpha_F > \dfrac{1}{2} \\ \Delta\{O, I\} & \text{if } \alpha_F = \dfrac{1}{2} \\ I & \text{if } \alpha_F < \dfrac{1}{2} \end{cases}$$

Now P1 must choose to which level of α_F to commit. The incumbent prefers to induce the entrant to stay out. Hence, $\alpha_F \geq 1/2$.

$$u_1(\alpha_F, BR_2(\alpha_F)) = u_1(\alpha_F, O) = 3\alpha_F + 4(1 - \alpha_F) = 4 - \alpha_F$$

which is strictly decreasing in α_F. Hence:

$$\alpha_F = \underset{\alpha_F \geq 0.5}{\arg\max}\, u_1(\alpha_F, BR_2(\alpha_F)) = 0.5$$

Thus, it is optimal for the incumbent to commit to $\alpha_F = 1/2$. At this α_F, the entrant will be indifferent between I and O, which in a Stackelberg equilibrium means he will choose O. Hence, the best pre-commitment for P1 is $\alpha_F = 1/2$ and he gets a payoff of 3.5.

$$\text{Stackelberg } = \{\alpha_F = 1/2, O\}$$

Side note: Let's finish calculating the set of dynamic equilibria in this game.
In a repeated game, Folk's theorem tells us that the possible payoffs will yield at least the minmax. **Note, however, that Folk's theorem is not applicable here because it relates to games where both players are patient, and here we have a SR player. We will hence focus in worst and best dynamic equilibria.**

The worst dynamic equilibria lies between the minmax and the static Nash, which in this exercise coincide. [4] To find the best-dynamic equilibria, we focus on the SR player's BR to the LR player's strategies, and we choose the best worst in the support. Notice that we have already derived the BR of the entrant when calculating the mixed Stackelberg: [5]

- If the LR player plays $\alpha_F \in [0, 1/2)$, we have seen that $BR_2(\alpha_F) = I$. The payoffs for the LR player if the entrant plays I are -2 and 2, the worst being -2.
- If the LR player plays $\alpha_F \in [1/2, 1]$, the $BR_2(\alpha_F) = O$. The payoffs for the LR player if the entrant plays O are 3 and 4, the worst being 3.

The best worst in the support is the highest between -2 and 3. Hence, the best dynamic equilibrium is 3.

The combination of all possible payoffs in this game with a SR and LR player is described by the line:

$$2\lambda + 3(1 - \lambda) \qquad \lambda \in [0, 1]$$

[4] To find the minmax, in this case it is enough to focus on pure strategies as there is a strictly dominant strategy.

[5] Notice how the cutoff point for α_F is defined in this BRs. This goes back to the discussion we had in the TA class: BR and equilibrium are two different notions. It is still a BR for player 2 to play In when alpha = 1/2. However, we are now explicitly looking for an equilibrium (the best dynamic one), which rules out BRs where $BR_2(\alpha_F = 1/2) = I$.

c. *What is the subgame perfect equilibrium if the game is repeated $T < \infty$ times?*

$\{G, I\}$ is the SPE of the game repeated $T < \infty$ as it is the only pure strategy equilibrium of the stage game.

d. *If the game is infinitely repeated, find a δ and strategies for both players such that the long-run player gets 3.*

Note that $\{F, O\}$ is the only history that guarantees a discounted average payoff of 3.

Consider the following grim trigger strategies:

$$a_I^t = \begin{cases} F & \text{if } \{F, O\} \text{ has always been played or } t = 1 \\ \text{G} & \text{otherwise} \end{cases}$$

$$a_E^t = \begin{cases} O & \text{if } \{F, O\} \text{ has always been played or } t = 1 \\ I & \text{otherwise} \end{cases}$$

We apply the single-deviation principle: for each history we write the augmented stage game and we check that there is no single profitable deviation.

There are two possible histories:

- $H_1 = \{F, O\}$ always.

  Remember that the **entrant** is a short run player who does not care about the future. In particular, when F is played he has no incentive to deviate. So in this history he has no profitable deviation.

  For the long run player the present value of sticking to F when out is played, is $V_F = \frac{3}{1-\delta}$, while the present value of any other strategy profile played is $V_G = \frac{2}{1-\delta}$, as if someone deviates they end up playing $\{G, I\}$ forever. So, the augmented game is:

		Ent	
Inc		Out	In
	fight	3+δV_f,0	-2+δV_f,-2
	give in	4+δV_g,0	2+δV_g,2

  For the entrant there is no profitable deviation from (F, O), while for the incumbent there is no profitable deviation iff:

  $$3 + \delta V_F \geq 4 + \delta V_G \qquad \delta \geq \frac{1}{2}$$

- $H_2 =$ at least once $\{F, O\}$ was not played.

  Now consider the second history where $\{F, O\}$ has not been played at a certain date. Then whatever players do the strategy profile still dictates to play $\{G, I\}$. Let's depict the augmented stage game and look at whether there is a profitable deviation from the strategy profile:

  We see that, for the entrant there is no profitable deviation ($2 > 0$), and neither does the incumbent as $2 + \delta V_G > -2 + \delta V_G$.

		Ent	
Inc		Out	In
	fight	$3+\delta V_g$,0	-$2+\delta V_g$,-2
	give in	$4+\delta V_g$,0	$2+\delta V_g$,2

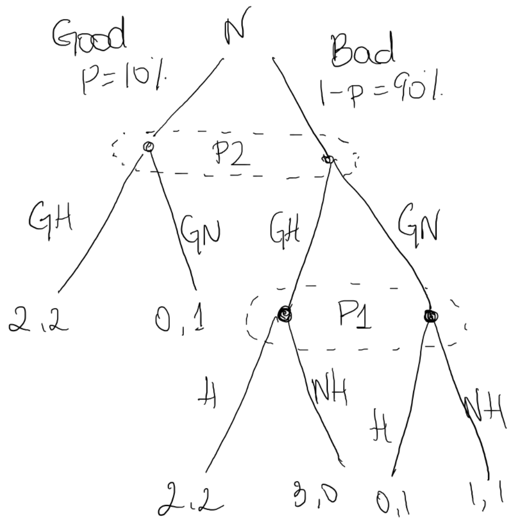
According to the single deviation principle, for $\delta \geq \frac{1}{2}$ players playing $\{a_I^t, a_E^t\} \quad \forall t$ as described above is a strategy profile that is a SPNE of the infinitely repeated game.

Exercise 5. Brazil or the U.S.?

A long-lived government faces a short-run representative government employee. The government must choose whether to honor pensions (H) or not (N). At the beginning of the period, times are either "good" or "bad." The probability times are "bad" is 90%. In good times, pensions are always honored. In bad times they are honored or not depending on the government decision. The employee is informed and observes (after the fact, at the end of the period) whether or not times are good or bad. The choice of the employee is to guess whether or not her pension will be honored (H) or (N). The payoff of the employee is the sum of two parts: 1 if the pension is honored, 0 if it is not; and 1 for guessing right, 0 for guessing wrong. So guessing right when the pension is honored gives 2, and so forth.

a. *Find the extensive and normal forms of the stage-game.*

   It is always a good idea to start with the extensive form, which will help us in building the normal forms.



To have the normal form representation, we first write the two games and then mix them according to their probability of occurrence. The game in good time occurs with 0.1 probability, while the bad state of the world happens 0.9 probability. So players face a game of expected outcomes.

Good (10%)	Emp		
Gov		GH	GN
	H	2,2	0,1
	N	2,2	0,1

Bad (90%)	Emp		
Gov		GH	GN
	H	2,2	0,1
	N	3,0	1,1

	Emp (2)		
Gov (1)		GH	GN
	H	2,2	0,1
	N	2.9,0.2	0.9,1

b. *For the long-run player, find the minmax, the static Nash, mixed precommitment and pure precommitment payoffs.*

- Minmax: the minmax is the lowest payoff to which the SR player can hold the LR. Call α_{GH} the probability that the employee guesses honour.

$$
\begin{aligned}
&u_1(H, \alpha_{GH}) = 2\alpha_{GH} \\
&u_1(N, \alpha_{GH}) = 2.9\alpha_{GH} + 0.9(1 - \alpha_{GH}) = 0.9 + 2\alpha_{GH} \\
&\min_{\alpha_{GH}\in[0,1]} \max\{2\alpha_{GH}, 0.9 + 2\alpha_{GH}\} = \min_{\alpha_{GH}\in[0,1]} 0.9 + 2\alpha_{GH} = 0.9
\end{aligned}
$$

  The minmax payoff is 0.9. [6]

- Static Nash: for the government to not honour is a dominant strategy. So, he plays N so that for the employee then GN is dominant strategy. So the static NE is $\{N, GN\}$ and its payoff is for the long-run player is 0.9.

- Pure precommitment: the long-run player can precommit to H or N. If he commits to H, then the employee's best response is GH and the government's payoff is 2. If he commits to N, the employees BR is GN and the government's payoff is 0.9. So, the government's pure precommitment strategy is H, and the payoff is 2.

- Mixed precomitment: Call α_H the probability of the government playing H. The government will set α_H to maximize its payoff according to the employee's BR function. The employee's indifference condition is:

$$
2\alpha_H + 0.2(1 - \alpha_H) = \alpha_H + 1 - \alpha_H \qquad \alpha_H = 4/9
$$

$$
BR_E(\alpha_H) = \begin{cases} GH & \text{if } \alpha_H > \frac{4}{9} \\ \Delta\{GH, GN\} & \text{if } \alpha_H = \frac{4}{9} \\ GN & \text{if } \alpha_H < \frac{4}{9} \end{cases}
$$

  Here again, we will use the tie-breaking rule. When indifferent, the SR player will choose the action that benefits the LR player. Note that, as discussed in the TA class, the BR of the SR player to $\alpha_H = \frac{4}{9}$ remains $\Delta\{GH, GN\}$, but we are stating that the equilibrium will be one where the SR chooses to play GH.

[6] Note that, in fact, N is a dominant strategy and so the employee cannot induce the government to play anything else but GN.

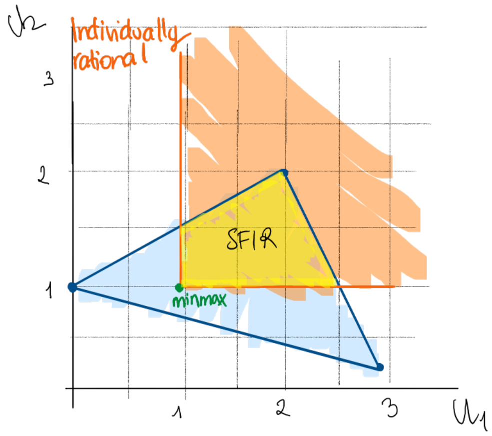
The government is always better off if the employee plays GH ($\alpha_H \geq 4/9$) and its payoff function is:

$$u_G\left(\alpha_H \geq \frac{4}{9}, BR_E(\alpha_H)\right) = u_G\left(\alpha_H \geq \frac{4}{9}, GH\right) = 2\alpha_H + (1-\alpha_H) * 2.9 = 2.9 - 0.9 * \alpha_H$$

This utility is decreasing in α_H, so that the government will aim at setting $\alpha_H = 4/9$. The mixed precommitment strategy equilibrium is $\{\alpha_H = 4/9, GH\}$, and the long-run player gets a payoff of $2.9 - 0.9 * 4/9 = 2.5$.

c. *Find the worst equilibrium for the long-run player, and describe in general terms the set of equilibrium payoffs for the long-run player.*

In a repeated game, Folk's theorem tells us that the possible payoffs will yield at least the minmax. We can draw the socially feasible and individually rational region to get a graphical intuition of where the possible payoffs lie.



Note, however, that Folk's theorem is not applicable here because it relates to games where both players are patient, and here we have a SR player. We will hence focus in worst and best dynamic equilibria. [7]

The worst possible equilibrium payoff lies between the minmax and the static Nash. In this example, as both yield the same payoff, the worst equilibrium for the long-run player will be 0.9, coming from the outcome at $\{N, GN\}$. To find the best-dynamic equilibria, we focus on the SR player's BR to the LR player's strategies, and we choose the best worst in the support. Notice that we have already derived the BR of the employee:

- If the LR player plays $\alpha_H \in [0, 4/9)$, we have seen that $BR_E(\alpha_H) = GN$. The payoffs for the LR player if the employee plays GN are 0 and 0.9, the worst being 0.
- If the LR player plays $\alpha_H \in [4/9, 1]$, the $BR_E(\alpha_H) = GH$. The payoffs for the LR player if the employee plays GH are 2 and 2.9, the worst being 2.

[7] Why is the best-dynamic equilibrium payoff below the mixed pre-commitment payoffs? Effectively, what this means is that a LR player may do better facing a LR opponent than a SR opponent. When faced with a LR opponent, a LR player can threat with future punishments if the decided strategy is not followed. These threats are meaningless when facing a SR opponent, and this forces the LR player to make concessions that invalidate the Folk's theorem.

The best worst in the support is the highest between 0 and 2. Hence, the best dynamic equilibrium is 2.

The combination of all possible payoffs in this game with a SR and LR player is described by the line:

$$\lambda(0.9) + (1-\lambda)2 \qquad \lambda \in [0,1]$$

d. *How patient must the government be to avoid catastrophe?*

Let's assume that catastrophe is $\{N, GN\}$, and the government wants to instead have $\{H, GH\}$. We will look for the δ such that the long-run equilibrium $\{H, GH\}$ can be sustained.

Define the following grim-trigger strategies:

$$a_G^t = \begin{cases} H & \text{if } \{H, GH\} \text{ has always been played or } t = 1 \\ N & \text{otherwise} \end{cases}$$

$$a_E^t = \begin{cases} GH & \text{if } \{H, GH\} \text{ has always been played or } t = 1 \\ GN & \text{otherwise} \end{cases}$$

The augmented game will depend on the history. Let's define the continuation payoffs for the government:

$$V_H = \frac{2}{1-\delta} \qquad\qquad V_N = \frac{0.9}{1-\delta}$$

1. $H_1 = \{H, GH\}$ has always been played.

   The augmented stage game for the first history is:

		Emp	
Gov		GH	GN
	H	2+δV_H,2	0+δV_N,1
	N	2.9+δV_N,0.2	0.9+δV_N,1

   Note that for the short-run player there is no profitable deviation ($2 > 1$). For the long-run player there is no profitable deviation for this history iff:

$$2 + \delta V_H \geq 2.9 + \delta V_N \qquad 2 + \delta\frac{2}{1-\delta} \geq 2.9 + \delta\frac{0.9}{1-\delta} \qquad \delta \geq \frac{0.9}{2}$$

2. $H_2 = \{H, GH\}$ was not played at some point.

   Whatever players do, the strategy prescribes that in the future both play $\{GN, N\}$ the government will always face V_N as the present discounted value of the continuation game. The augmented game is:

   The short-run player has no profitable deviation, as $1 > 0.2$. There is no δ for the government such that $0.9 + \delta V_N > 0 + \delta V_N$, so he will also stick to the grim-trigger strategy.

		Emp	
		GH	GN
Gov	H	2+δV_N,2	0+δV_N,1
	N	2.9+δV_N,0.2	0.9+δV_N,1

Exercise 6. The Folk Theorem

For each of the following simultaneous move games, find the static Nash equilibria, and give an accurate sketch of the socially feasible individually rational region.

a. Static Nash: D is dominant strategy, R is dominant strategy, so $PSNE = \{D, R\}$.

	L	**R**
U	4,3	0,7
D	5,0	1,2

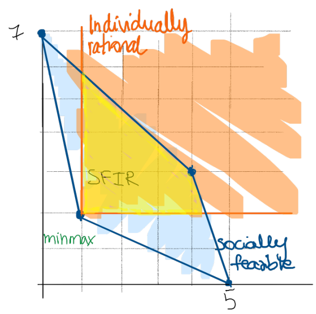
Let's look for the min max now. For player 1, when player 2 plays L with α_L probability. [8]

$$u_1(U, \alpha_L) = 4\alpha_L$$
$$u_1(D, \alpha_L) = 1 + 4\alpha_L$$
$$\min_{\alpha_L \in [0,1]} \max\{4\alpha_L, 1 + 4\alpha_L\} = \min_{\alpha_L \in [0,1]} 1 + 4\alpha_L = 1$$

For player 2, when player 1 plays U with α_U probability.

$$u_2(\alpha_U, L) = 3\alpha_U$$
$$u_2(\alpha_U, R) = 2 + 5\alpha_U$$
$$\min_{\alpha_U \in [0,1]} \max\{3\alpha_U, 2 + 5\alpha_U\} = \min_{\alpha_U \in [0,1]} 2 + 5\alpha_U = 2$$

The minmax is for P1 is 1 while for P2 is 2 (payoffs of $\{D, R\}$).

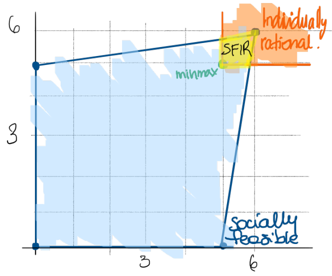


b. Static Nash: U is dominant for both 1 and 2, so $PSNE = \{U, L\}$.

The minmax for P1 is 5 (he plays U and P2 minimize's him playing R), and the minmax for P2 is 5 (he plays L and P1 plays D).

[8]Because there are strictly dominant strategies, we could restrict our analysis of the min max to pure strategies.

	L	**R**
U	6,6	5,0
D	0,5	0,0



Exercise 7. Equilibrium in a Repeated Game

Consider the simultaneous move stage game. Consider the "grim" strategy of playing U in period one, playing U as long as both players have played U in the past, and playing D otherwise. For what discount factors δ do these strategies form a subgame perfect equilibrium?

	U	**D**
U	1,1	-1, 100
D	100, 1	0,0

Note the game is symmetric what holds for one player holds also for the other. The grim-trigger strategy is:

$$a_i^t = \begin{cases} U & \text{if } \{U,U\} \text{ has always been played or } t=1 \\ D & \text{align} \end{cases} \qquad \forall i = \{1,2\}$$

Define the continuation payoffs as:

$$V_U = \frac{1}{1-\delta} \qquad V_D = 0$$

- Augmented stage game for the history $H_1 = \{U,U\}$:

		P2	
P1		U	D
	U	1+δV_u,1+δV_u	-1+δV_D,100+δV_D
	D	100+δV_D,-1δV_D	0+δV_D,0+δV_D

There is no profitable deviation if:

$$1 + \delta\frac{1}{1-\delta} \geq 100 \qquad \delta \geq \frac{99}{100}$$

- Augmented game for the case where $H_2 = \{U,U\}$ has not always been played:

There is no profitable deviation $\forall \delta$.

		P2	
P1		U	D
	U	1+δV_D,1+δV_D	-1+δV_D,100+δV_D
	D	100+δV_D,-1+δV_D	0+δV_D,0+δV_D

So the strategies $\{a_1^t, a_2^t\}$ constitute a SPNE $\forall t$ if $\delta \geq 99\%$.