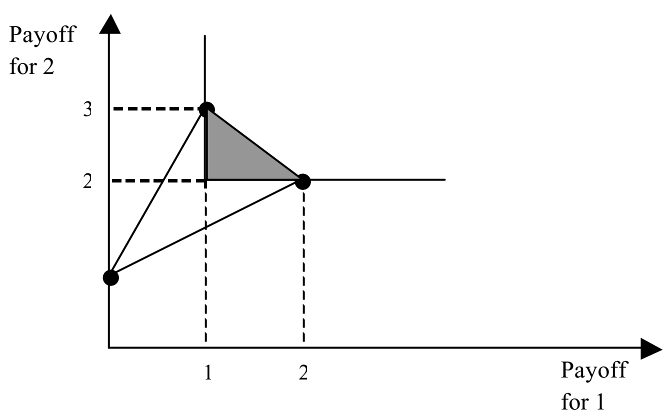
a)

	L	R
U	0,1	1*,3*
D	2*,2*	0,1

Nash Equilibria in pure strategies: {(D,L);(U,R)}
Nash Equilibrium in mixed strategies:

	L **α**	R **(1-α)**
U **β**	0,1	1*,3*
D **(1-β)**	2*,2*	0,1

$1-\alpha = 2\alpha \Rightarrow \alpha=1/3$
$\beta+2(1-\beta) = 3\beta+(1-\beta) \Rightarrow \beta=1/3$
Therefore the mixed strategy NE is $\{\frac{1}{3}U+\frac{2}{3}D, \frac{1}{3}L+\frac{2}{3}R\}$
Both Pure strategy NE are Pareto Efficient.
The mixed strategy NE is Pareto Inefficient since the expected payoffs are (2/3,5/3).
No weakly or strictly dominated strategies are played.

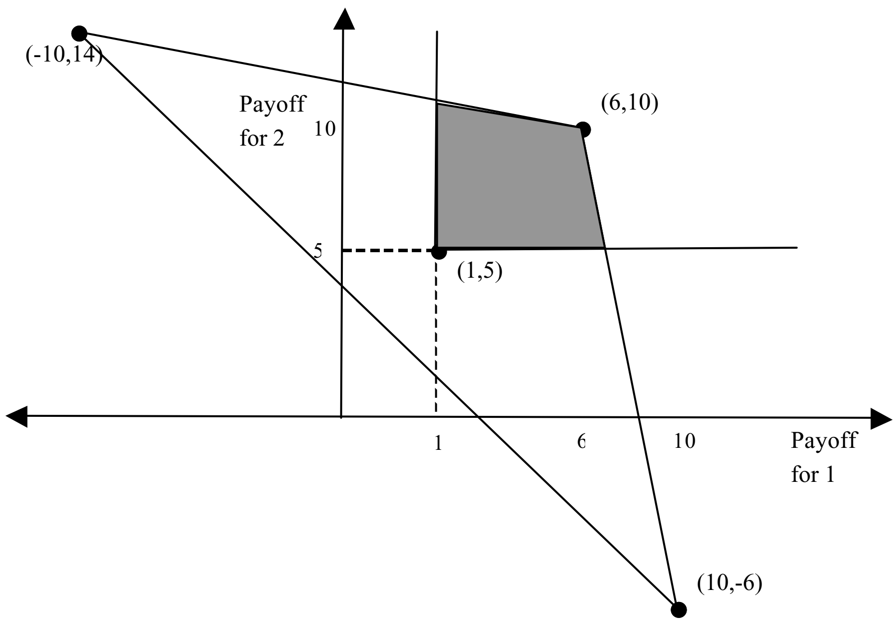


Minmax for player 1: 1
Minmax for player 2: 2

b)

	L	R
U	6,10	-10,14*
D	10*,-6	1*,5*

Nash Equilibrium in pure strategies: {(D,R)}
NO Nash Equilibrium in mixed strategies since D and R are dominant strategies.
(D,R) is Pareto Inefficient.
No weakly or strictly dominated strategies are played.

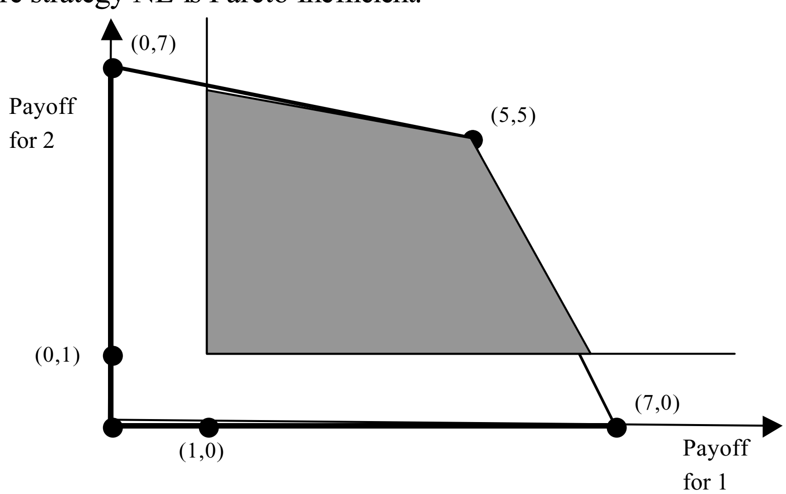


Minmax for player 1: 1
Minmax for player 2: 5

c)

	L	C	R
U	6,6	0,7*	1*,0
M	7*,0	5*,5*	1*,0
D	0,1*	0,1*	0,0

Nash Equilibria in pure strategies: {(M,C)}
Pure strategy NE is Pareto Inefficient.



Minmax for player 1: 1
Minmax for player 2: 1

2)

	A	B
A	7,6*	2,5
B	8*,2	3*,3*

Consider Grim strategies for both players:

Play A in first period
Play A as long as no deviation from AA took place in the past.
Play B otherwise forever.

$$U_1(A) = \frac{7}{1-\delta}$$

$$U_1(B) = 8 + 3\frac{\delta}{1-\delta}$$

$$8+3\frac{\delta}{1-\delta} \leq \frac{7}{1-\delta} \Leftrightarrow \frac{1}{5} \leq \delta$$

Therefore any δ greater or equal to 0.2 would support AA as the SGPE (NE) of the repeated game.
The proposed strategies are SGP because the punishment to deviations implies playing the NE of the stage game (which is SGP).

Minmax for 1: 3
Minmax for 2: 3

Folk Theorem stated that if the discount factor δ is sufficiently close to 1, all points in the socially feasible individually rational region are SGPE of the repeated game.

3. Long Run versus Short Run

Maria must decide whether to bring her broken scooter to the dealer or to ScooterRepairsRus (SRU for short). If she brings it to the dealer it is costly to repair, but the scooter will work properly after the repair. This will give her a net utility of zero. On the other hand, if she brings the scooter to SRU, SRU may either fix the scooter cheap (worth a utility of 1), or rip her off (worth a utility of –1). SRU gets 0 if Maria brings the scooter to the dealer; 1 for fixing the scooter cheap and 7 for ripping her off.

a) Find the extensive and normal form of this game.

b) What pure strategy Nash equilibria are in the stage game; which are subgame perfect?

c) What is the Stackelberg equilibrium of the stage game in which SRU moves first?

d) Suppose that this stage game is repeated: SRU is infinitely lived with discount factor equal to δ and there is a sequence of short-lived consumers (Maria and her friends). Propose a strategy and a discount factor δ such that in equilibrium players end up playing the Stackelberg equilibrium.

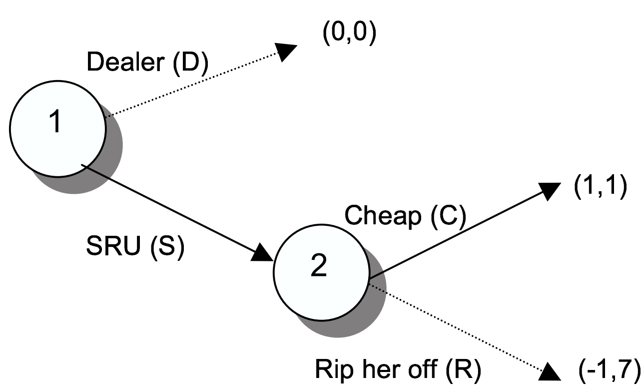
e) What difference would reputation make in the repeated case?

Answers

a) and b)

Player 1: Maria
Player 2: SRU

Extensive form with subgame perfect choices marked with dashed lines



Subgame perfect equilibrium: (D,R)

Normal form with best response correspondence and Nash equilibria marked

	C	R
D	0,0*	0*,0*
S	1*,1	-1,7*

Nash equilibrium: (D,R)

c) Stackelberg equilibrium:

	C	R
D	0,0	0*,0
S	1*,1	-1,7

Fix the scooter cheap: 1

Rip her off: 0

Stackelberg equilibrium is to fix the scooter cheap. Payoff equal to 1.

d) Strategy and a discount factor δ such that in equilibrium players end up playing the Stackelberg equilibrium:

Strategy:

Use the grim strategies: C(or S) as long as SC in every past period, otherwise DR (the static Nash equilibrium).

Discount Factor:

Payoff with each strategy:

Cooperate (always C)

1

Deviate from cooperation (R)

$7(1-\delta)+0\delta$

In order to obtain the stackelberg equilibrium the following must hold:

$1 \geq 7(1-\delta)$

$\delta \geq 6/7$

e) With reputation a sufficiently patient long run player (SRU) can get close to the Stackelbeg payoff.

Answer Key Q.4, Decision Analysis

Matias Iaryczower, June 5, 2002

V(U) = 10 with probability Pr(U) = 0.6

V(D) = -10 with probability Pr(D)= 0.4

Then E[U(Buy)] = (0.6).10 + (0.4).(-10) = 2 > E[U(Out)] = 0, so buying the stock directly is better than not buying it directly.

Now consider the option of buying a tip from the broker at price p. The broker sends you a signal $S \in \{S_U, S_D\}$, with

$Pr(S_U/U) = 0.8$ (so $Pr(S_D/U) = 0.2$), and

$Pr(S_D/D) = 0.9$ (so $Pr(S_U/D) = 0.1$)

Then using Bayes' law, we can compute

$$\Pr(U/S_U) = \frac{\Pr(S_U/U)\Pr(U)}{\Pr(S_U/U)\Pr(U) + \Pr(S_U/D)\Pr(D)} = \frac{(0.8)(0.6)}{(0.8)(0.6)+(0.1)(0.4)} = \frac{12}{13}$$

$$\Pr(U/S_D) = \frac{\Pr(S_D/U)\Pr(U)}{\Pr(S_D/U)\Pr(U) + \Pr(S_D/D)\Pr(D)} = \frac{(0.2)(0.6)}{(0.2)(0.6)+(0.9)(0.4)} = \frac{1}{4}$$

Hence,

$E[U(Buy)/S_U] = (12/13) .10 + (1/13) . (-10) – p = (110/13) – p$

$E[U(Buy)/S_D] = (1/4) .10 + (3/4) . (-10) – p = -5 – p$

$E[U(Out)/S_U] = E[U(Out)/S_D] = - p$

Having bought the tip, you should buy the stock when you receive signal S_U , since $E[U(Buy)/S_U] = (110/13) – p > E[U(Out)/S_U] = - p$, and you shouldn't buy the stock when you receive signal S_D, since $E[U(Out)/S_D] = - p > E[U(Buy)/S_D] = -5 – p$.

What is the expected utility of buying the tip ? As we computed above, $Pr(S_U)$ = (0.8).(0.6) + (0.1).(0.4) = 0.52, so that $Pr(S_D)$ = 0.48. Then

$$E[U(Tip)] = Pr(S_U).E[U(Buy)/S_U] + Pr(S_D).\ E[U(Out)/S_D]$$
$$= (0.52).\ [(110/13) - p] + (0.48).\ (-p) = 4.4 - p$$

Then you should buy the tip if E[U(Tip)] > Max{E[U(Buy)], E[U(Out)]} = E[U(Buy)]. That is, you should buy the tip if $4.4 - p > 2$, or if $p < 2.4$. If $p \geq 2.4$ you shouldn't buy the tip, but buy the stock directly.

5. Cournot with Uncertain Cost:

1. Bayesian Nash Equilibrium:

   (a) Conditional Probability:
   According to the description of the question, the probability distribution with respect to marginal costs is the following:

	$MC_2 = 1$	$MC_2 = 3$
$MC_1 = 1$	0.2	0.2
$MC_1 = 3$	0.2	0.4

   Hence, given firm 1 has high marginal cost, the conditional probability that firm 2 has low cost is:

$$\begin{aligned} P(MC_2 = 1 \mid MC_1 = 3) &= \frac{P(MC_2 = 1, MC_1 = 3)}{P(MC_1 = 3)} \\ &= \frac{0.2}{0.6} \\ &= \frac{1}{3} \end{aligned}$$

   Similarly, we can derive all the conditional probabilities:

$$\begin{aligned} P(MC_{-i} = 1 \mid MC_i = 1) &= \frac{1}{2} \\ P(MC_{-i} = 3 \mid MC_i = 1) &= \frac{1}{2} \\ P(MC_{-i} = 1 \mid MC_i = 3) &= \frac{1}{3} \\ P(MC_{-i} = 3 \mid MC_i = 3) &= \frac{2}{3} \end{aligned}$$

   (b) Best-Response Function:
   Let x^1 be the equilibrium output of firms with low marginal cost. Let x^3 be the equilibrium output of firms with high marginal cost. If firm i has low marginal cost, $MC_i = 1$, its optimal problem will be the following:

$$\begin{aligned} \max_{x_i} \pi_i &= [17 - (x_i + (\frac{1}{2}x^1 + \frac{1}{2}x^3))]x_i - x_i \\ &= [16 - (x_i + (\frac{1}{2}x^1 + \frac{1}{2}x^3))]x_i \\ \frac{\partial \pi_i}{\partial x_i} &= 16 - (x_i + \frac{1}{2}x^1 + \frac{1}{2}x^3) - x_i \equiv 0 \end{aligned}$$

Since in equilibrium $x_i = x^1$, the best-response function of a firm with low marginal cost is:

$$\frac{5}{2}x^1 + \frac{1}{2}x^3 = 16 \tag{1}$$

If firm j has high marginal cost, $MC_j = 3$, its optimal problem will be the following:

$$\begin{aligned}
\max_{x_j} \pi_j &= [17 - (x_j + (\frac{1}{3}x^1 + \frac{2}{3}x^3))]x_j - 3x_j \\
&= [14 - (x_j + (\frac{1}{3}x^1 + \frac{2}{3}x^3))]x_j \\
\frac{\partial \pi_j}{\partial x_j} &= 14 - (x_j + \frac{1}{3}x^1 + \frac{2}{3}x^3) - x_j \equiv 0
\end{aligned}$$

Since in equilibrium $x_j = x^3$, the best-response function of a firm with high marginal cost is:

$$\frac{1}{3}x^1 + \frac{8}{3}x^3 = 14 \tag{2}$$

Solving equations (1) and (2), we get the equilibrium outputs:

$$\begin{aligned}
(x^1, x^3) &= (\frac{214}{39}, \frac{178}{39}) \\
&\doteq (5.487, 4.564)
\end{aligned}$$

(c) Bayesian Nash Equilibrium:

- With probability 0.2, both firms are low-cost firms.
  The output for each firm is $\frac{214}{39}$ ($\doteq$ 5.487). The industry output is $\frac{428}{39}$ ($\doteq$ 10.974).
- With probability 0.4, both firms are high-cost firms.
  The output for each firm is $\frac{178}{39}$ ($\doteq$ 4.564). The industry output is $\frac{356}{39}$ ($\doteq$ 9.128).
- With probability 0.4, one of the firms is low-cost and the other is high-cost.
  The output for the low-cost firm is $\frac{214}{39}$ ($\doteq$ 5.487). and the output for the high-cost firm is $\frac{178}{39}$ ($\doteq$ 4.564). The industry output is $\frac{392}{39}$ ($\doteq$ 10.051).

2. Nash Equilibrium with Complete Information:
Let x_1 be firm 1's output level; MC_1 be firm 1's marginal cost. Let x_2 be firm 2's output level; MC_2 be firm 2's marginal cost. Let x be the industry output. When the information is complete, both firms know their own costs and their rivals' costs. So each firm maximizes its profit given the other firm's output level.

(a) $MC_1 = 1$ and $MC_2 = 1$:
Firm 1's best-response function is

$$2x_1 + x_2 = 16$$

which is derived by the following:

$$\begin{aligned} \max_{x_1} \pi_1 &= [17 - (x_1 + x_2)]x_1 - x_1 \\ &= [16 - (x_1 + x_2)]x_1 \\ \frac{\partial \pi_1}{\partial x_1} &= 16 - 2x_1 - x_2 \equiv 0 \end{aligned}$$

Since both firms have the same profit function, by symmetry $x_1 = x_2 = \frac{16}{3}$. Hence the industry output is $x = \frac{32}{3} \doteq 10.67$.

(b) $MC_1 = 3$ and $MC_2 = 3$:
Firm 1's best-response function is

$$2x_1 + x_2 = 14$$

which is derived by the following:

$$\begin{aligned} \max_{x_1} \pi_1 &= [17 - (x_1 + x_2)]x_1 - 3x_1 \\ &= [14 - (x_1 + x_2)]x_1 \\ \frac{\partial \pi_1}{\partial x_1} &= 14 - 2x_1 - x_2 \equiv 0 \end{aligned}$$

Since both firms have the same profit function, by symmetry, $x_1 = x_2 = \frac{14}{3}$. Hence the industry output is $x = \frac{28}{3} \doteq 9.33$.

(c) $MC_1 = 1$ and $MC_2 = 3$:
Firm 1's best-response function is

$$2x_1 + x_2 = 16 \tag{3}$$

Firm 2's best-response function is

$$x_1 + 2x_2 = 14 \tag{4}$$

Solving equations (3) and (4), we get $x_1 = 6$ and $x_2 = 4$. Hence the industry output is $x = 10$.

(d) $MC_1 = 3$ and $MC_2 = 1$:
The industry output level is the same as the previous case, i.e. $x = 10$.